Ambiguity under Growing Awareness*

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Abstract

In this paper, we study choice under growing awareness in the wake of new discoveries. The decision maker's behavior is described by two preference relations, one before and one after new discoveries are made. The original preference admits a *subjective expected utility* representation. As awareness grows, the original decision problem expands and so does the state space. Therefore, the decision maker's original preference has to be extended to a larger domain, and consequently the new preference might exhibit ambiguity aversion. We propose two consistency notions that connect the original and new preferences. *Unambiguity Consistency* requires that the original states remain unambiguous while new states might be ambiguous. This provides a novel interpretation of ambiguity aversion as a systematic preference to bet on old states than on newly discovered states. *Likelihood Consistency* requires that the relative likelihoods of the original states are preserved. Our main results axiomatically characterize a *maxmin expected utility* (MEU) representation of the new preference that satisfies the two consistency notions. We also extend our model by allowing the initial preference to be MEU, and characterize reverse full-Bayesiansim, which is an extension of the reverse Bayesianism of Karni and Vierø (2013) to MEU preferences.

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1 Introduction

When modeling choice behavior under uncertainty, economists take for granted that the description of the underlying decision problem – including the states of nature, actions and consequences – is fixed. However, in many real life situations, a decision maker (henceforth, DM) makes new discoveries that change the decision problem. New scientific insights, novel technologies, new medical treatments, new financial instruments, or new goods emerge on an almost daily basis. Such discoveries might reveal contingencies of which the DM was unaware.[1] As awareness grows, the DM's universe (i.e., state space) expands and this might affect her preferences. In this paper, we explore how the DM's beliefs and tastes evolve.

We provide a theory of choice under growing awareness in which a subjective expected utility (SEU) preference (Anscombe and Aumann, 1963) extends to a maxmin expected utility (MEU) preference (Gilboa and Schmeidler, 1989). While the extended preference captures ambiguity aversion, it inherits some properties of the initial SEU preference. Our theory provides a novel interpretation of ambiguity aversion. In particular, ambiguity arises because the DM treats new and old states differently; there is no exogenous information about states. In contrast, in the Ellberg experiments exogenous information about states is provided and ambiguity arises since the DM treats states with known and unknown probabilities differently.

To illustrate changes in beliefs due to growing awareness, consider a patient who suffers from a disease and needs to choose an appropriate treatment. There are two standard treatments, A and B. Each treatment leads to one of two possible outcomes: a success or a failure. The patient believes that each treatment is successful with probability $\frac{1}{2}$.

Suppose now that the patient consults her doctor and discovers that there is a new treatment N, which may be successful or not. Since this treatment is new, the patient faces a new decision problem. How does she extend her original beliefs to evaluate the new treatment? First, the patient needs to form new beliefs about the conceivable outcomes of the new treatment in order to evaluate it. Second, the new discovery might causes the patient to reevaluate her original beliefs regarding the outcomes of the standard treatments A and B.

Given the discovery of treatment N, the patient's original preferences might change fundamentally. In particular, she may not be able to come up with a unique probability that treatment N will be successful, and therefore she may be cautious about the novel treatment. To discipline the effect of new discoveries, we consider two consistency notions between the patient's behavior before and after the discovery.

Behaviorally, our consistency notions can be described by the way the standard treatments are evaluated after awareness has changed. Our first consistency notion requires that treatments A and B are still evaluated with respect to a (unique) probability measure over old states, yet the measure might change. Under this consistency notion, the values of A and B might change

[1] For Schipper (2014a, p.1), "*unawareness refers to the lack of conception rather than the lack of information.*" Under lack of information, the DM does not know which conceivable states may occur, whereas under lack of conception, she cannot even conceive that there may be other conceivable states which will determine her payoffs.

after the discovery. However, our second consistency notion requires that the ranking over the old treatments does not change as awareness grows. In general, the two consistency notions are independent.[2] Our goal is to axiomatically characterize each consistency notion by connecting the patient's initial and new preferences.

To study growing awareness formally, we introduce a general framework that can cover other widely studied approaches to model growing awareness such as Karni and Vierø (2013) (KV, henceforth) and Heifetz et al. (2006, 2008, 2013). Although our framework is more general than KV's, it is useful to contrast our approaches. They construct state spaces explicitly by invoking the approach of Schmeidler and Wakker (1990) and Karni and Schmeidler (1991): a state specifies the unique consequence that is associated with every act. Within this approach, KV have introduced an elegant theory of choice under growing awareness called *reverse Bayesianism*. They focus on SEU preferences and characterize the evolution of probabilistic beliefs in the decision theoretic framework of Anscombe and Aumann (1963). However, under reverse Bayesianism, growing awareness does not affect the SEU form of preferences and thus the theory precludes ambiguity.

In contrast, our theory allows the DM's behavior to change fundamentally as awareness grows.[3] While being originally a SEU maximizer, the DM might become ambiguity averse in an expanded universe. More specifically, the DM's behavior is described by two preference relations, one before and one after a discovery is made. The initial preference takes the SEU form. One can think of this assumption as follows: the DM is relatively familiar with the original decision problem and therefore she has come up with a (unique) probability measure over the states. As awareness grows, the extended preference admits a MEU representation, capturing ambiguity in an expanded state space.

Our main results behaviorally characterize the evolution of an original SEU preference to a new, extended MEU preference under both consistency notions. The first consistency notion, called *Unambiguity Consistency*, requires that the new events that correspond to the old states are revealed to be unambiguous by the new MEU preference; only the new states may be ambiguous.

An extended MEU preference that satisfies Unambiguity Consistency is characterized by a novel axiom called *Negative Unambiguity Independence* (NUI) (see Theorem 1). It states that the DM can hedge against the ambiguity of the new acts (e.g., the new treatment N). However, mixing them with the old acts (e.g., the standard treatments A or B) cannot be used as a hedging strategy.

Under Unambiguity Consistency, new discoveries might affect the DM's ranking over the old acts since her probabilistic beliefs (as well as her risk preference) might change. For instance, after the discoveries of the new treatment N, the patient might believe that success under treatment A is more likely than under treatment B, leading her to strictly prefer the old treatment A over B.

[2] However, in our patient example the second consistency notion implies the first one.

[3] There is empirical evidence suggesting that individuals' awareness and ambiguity are related. For instance, Giustinelli and Pavoni (2017) ask Italian middle schoolers about their likelihoods to successfully graduate from different high school tracks. The authors find that students who were initially unaware of some school tracks (but learn their existence during the survey) perceive significant ambiguity about the success of alternative curricula that have these tracks. Related to our patient story, there is a growing body of evidence reporting people's ambiguity averse attitude when they face new medical tests and treatments (e.g., see Han et al. (2009) and Taber et al. (2015)).

Our second consistency notion, called *Likelihood Consistency*, requires that the new extended preference maintains the relative likelihoods of the original states. Preserving the original likelihoods might be reasonable in choice situations where the original preference is supported by hard facts or objective information. Likelihood Consistency is characterized by *Binary Awareness Consistency* (BAC), which requires that the DM's ranking of old acts (standard treatments A and B) is not affected by growing awareness (Theorem 2).[4]

Due to growing awareness the state space expands. We are particularly interested in two types of state space expansions, *refined expansion* and *genuine expansion.* Under refined expansions, the DM becomes aware of "finer descriptions" of the original states. However, under genuine expansions, the DM becomes aware of "completely new" states.[5] Axiomatic characterizations of the two consistency notions do not distinguish these two separate cases explicitly. Regardless, the two cases can be behaviorally disentangled, which could be difficult when both preferences are SEU. When the new treatment N is discovered, the state space is refined: each original state is extended by indicating whether treatment N leads to a success or a failure. In this context, our theory implies that the old treatments A and B are unambiguous acts while the new treatment N is ambiguous. Consequently, an ambiguity averse DM tends to prefer the old, unambiguous acts to the newly discovered ones.

However, in the context of genuine expansion of the state space ambiguity aversion will be exhibited differently. Suppose that the patient discovers that the standard treatments A and B might cause a health complication. In this case, the original state space is extended by completely new states indicating whether treatments A and B lead to the health complication or not (i.e., genuine expansion). If the patient perceives ambiguity about the completely new states, then the standard treatments become ambiguous acts. Therefore, the patient prefers mixtures between the old treatments A and B over A (and B).

At a fundamental level, our theory provides a novel interpretation of the widely-studied ambiguity phenomenon. Typically, as in the classical Ellsberg experiments, ambiguity is exogenously created. That is, subjects are informed about exogenous probabilities for some events in a given state space, and for other states such information is missing. The task is to elicit subjects' attitudes towards ambiguity. A systematic preference for betting on known probability events rather than betting on unknown probability events is understood as aversion towards the exogenous ambiguity.

In our theory, the DM perceives ambiguity about states of which she was originally unaware. In other words, an expanding universe can be seen as a "source" of ambiguity. As awareness grows, the DM cannot extend her old subjective belief uniquely so that her new beliefs are represented by a set of probability measures. Therefore, in our theory ambiguity aversion is displayed differently as a preference for betting on old, familiar states rather than betting on the newly discovered states.[6]

[4]Notice that Unambiguity Consistency is trivially satisfied in the setup of KV. The extended preference is SEU in a special case where all the newly discovered states are unambiguous. In this case, Theorem 2 provides an alternative characterization (interpretation) of reverse Bayesianism of KV.

[5]For example, in the KV framework, the discovery of acts leads to refined expansions while the discovery of consequences lead to genuine expansions.

[6]Indeed, Daniel Ellsberg describes ambiguity as a much broader phenomenon than a comparison between known

To show that Likelihood Consistency and Unambiguity Consistency can be extended to the setting where one MEU preference evolves to another MEU preference, we extend our model by allowing the initial preference to be a MEU preference (see Section 5). This extension also illustrates that our framework can be extended to settings with more than two periods. Under the reverse Bayesianism of KV, the initial prior is the Bayesian update of the extended prior. Interestingly, under our extension of Likelihood Consistency, called *reverse full-Bayesianism*, the initial set of priors is the full-Bayesian (i.e., prior-by-prior) update of the extended set of priors. Hence, our reverse full-Bayesianism is an extension of the reverse Bayesianism of KV to MEU preferences.

The rest of the paper is organized as follows. Section 2 presents the basic setup and illustrates how new discoveries expand the original state space. In Section 3, we discuss the SEU and MEU representations of the original and new preferences, and provide our definitions of consistent evolution of beliefs. In Section 4, we provide representation theorems that characterize our two consistency notions. In Section 5, we consider the case where the initial preference is MEU. In Sections 4.4 and 5.3, we study dynamic consistency of MEU preferences. In Section 6, we study a parametric version of our MEU representation. A brief overview of the literature on choice under (un)awareness is provided in Section 7. The proofs are collected in Appendix A.

2 Decision Problems

In this section, we demonstrate how a DM's decision problem expands as awareness grows.

Let S be a nonempty, finite set of states. The elements of S represent all the possible resolutions of uncertainty. Let C be a nonempty, finite set of consequences. Denote by $\Delta(C)$ the set of lotteries on C, i.e., functions $p : C \to [0,1]$ such that $\sum_{c\in C} p(c) = 1$. Objects of choice are the so-called Anscombe-Aumann acts; i.e., mappings from states to lotteries. Let $\hat{F} \equiv \{f : S \to \Delta(C)\}$ be the set of all acts. A collection $\mathcal{D} \equiv \{S, C, \hat{F}\}$ describes a choice problem under uncertainty. The DM's behavior is modeled via a preference relation on $\hat{F}$, denoted by $\succcurlyeq_{\hat{F}}$. As usual, $\succ_{\hat{F}}$ and $\sim_{\hat{F}}$ are the asymmetric and symmetric parts of $\succcurlyeq_{\hat{F}}$, respectively.

Suppose the DM initially faces a choice problem $\mathcal{D}$, called the *original decision problem*. Our primary goal is to examine how the DM's behavior might change as her awareness grows due to new discoveries. To this end, the original decision problem needs to be reformulated to incorporate growing awareness. We model choice situations in which $\mathcal{D}$ expands to a new choice problem $\mathcal{D}_1 \equiv \{S_1, C_1, \hat{F}_1\}$ where $|S| < |S_1|$, $C \subseteq C_1$, and $\hat{F}_1 \equiv \{f : S_1 \to \Delta(C_1)\}$. We call $\mathcal{D}_1$ *the extended decision problem*. It captures the DM's increased awareness since the state space and the sets of acts and consequences "expand." We denote by $\succcurlyeq_{\hat{F}_1}$ the *extended preference relation* on the extended set of acts $\hat{F}_1$ in $\mathcal{D}_1$. We investigate how the DM's original preference relation $\succcurlyeq_{\hat{F}}$ evolves to $\succcurlyeq_{\hat{F}_1}$.

The new state space S_1 depicts the expansion of the original states space S due to new discoveries. We need additional notations to discuss the relationships between S and S_1 and $\hat{F}$ and $\hat{F}_1$.

and unknown probabilities. In his words, ambiguity refers to "*a quality depending on the amount, type, reliability and 'unanimity' of information, and giving rise to one's degree of 'confidence' in an estimate of relative likelihoods (Ellsberg, 1961, p.657).*" In our setup, ambiguity might arise even if there are no objective or exogenous probabilities.

Let $E : S \to 2^{S_1} \setminus \{\emptyset\}$ be a mapping such that $E_s \cap E_{s'} = \emptyset$ for any $s, s' \in S$. Here E_s is called the *corresponding event* in S_1 of an original state $s \in S$. Let $S_1^R \equiv \bigcup_{s \in S} E_s$ and $S_1^N \equiv S_1 \setminus S_1^R$. Then E_s can be understood as the set of new states in S_1^R that "refine" s. We call S_1^R a set of *refinements* of S and S_1^N a set of *genuine extensions* of S.

We casually use phrases "new" and "old" acts. An act f in the $\hat{F}_1$ is an *old act* if, for any $s \in S$ and $s^1, \tilde{s}^1 \in E_s$, $f(s^1) = f(\tilde{s}^1) \in \Delta(C)$. Otherwise, we say $f \in \hat{F}_1$ is a *new act*. Therefore, any old act takes the following form:

$$(q_s\, E_s)_{s \in S} \cup g = \begin{cases} q_s & \text{if } s^1 \in E_s; \\ g(s^1) & \text{if } s^1 \in S_1^N. \end{cases} \tag{1}$$

That is, $(q_s\, E_s)_{s \in S} \cup g$ assigns q_s to states in E_s and g to states in S_1^N. Note that for any $f \in \hat{F}$ and $g \in \hat{F}_1$, we can construct an old act $(f(s)\, E_s)_{s \in S} \cup g$ in $\hat{F}_1$. Therefore, we can interpret that the set of acts $\hat{F}$ "expands" to $\hat{F}_1$.

2.1 Refined and Genuine Expansions

We are particularly interested in two special cases of the state space expansion. In the first case, the DM becomes aware of "finer descriptions" of the original states. That is, the original state space S expands to $S_1 = S_1^R$. In other words, the collection of events $\{E_s\}_{s \in S}$ forms a partition of the expanded state space S_1; i.e., $S_1 = S_1^R = \bigcup_{s \in S} E_s$. For example, suppose that originally a DM is aware that each day may be either rainy (r) or sunny (s); i.e., $S = \{r, s\}$. When she becomes aware that temperature matters and each day maybe either hot (h) or cold (c), each original state is refined by the additional information, i.e., $S_1 = \{rh, rc, sh, sc\}$ where $E_r = \{rh, rc\}$ and $E_s = \{sh, sc\}$. In this case, we say S_1 is a **refined expansion** of S.

In the second case, the DM becomes aware of "completely new" states. That is, the original state space S expands to $S_1 \equiv S_1^R \cup S_1^N$ such that $|S_1^R| = |S|$ and $S_1^N \neq \emptyset$. In other words, E_s becomes a state in S_1^R that corresponds to $s \in S$. For instance, the DM becomes aware that there may be a tornado (t); i.e., $S_1 = \{r, s, t\}$ where $E_r = \{r\}$ and $E_s = \{s\}$. In this case, we say S_1 is a **genuine expansion** of S.

Our setup captures other widely studied approaches to model growing awareness such as Karni and Vierø (2013, 2015), Heifetz et al. (2006, 2008, 2013), and Dietrich (2018). However, none of these papers studies ambiguity. Heifetz et al. (2006, 2008, 2013) provide an elegant model of growing awareness which is accommodated via a lattice of disjoint state spaces.[7] State spaces are ordered by richness of the vocabularies used to describe states. Each state space corresponds to one awareness level. As a DM discovers more "expressive" descriptions than the descriptions of the original state space, she discovers a richer state space associated with a higher awareness level.

[7] As Dekel et al. (1998) show, the standard approach used to model private information – via a state space with a partition of it – cannot capture unawareness. To model unawareness, the state space approach has to be augmented with a structure that accommodates an expansion of the original state space, referring to growing awareness.

Discoveries of more expressive vocabularies lead to refined expansions of the original state space.[8] Dietrich (2018) extends the SEU theorem of Savage (1954) under growing awareness. He also studies refined and genuine expansions of the original state space. Below we explain the framework of Karni and Vierø (2013) (henceforth, KV) in more detail to illustrate how the original state space expands.

KV-approach. In KV, growing awareness is directly linked to discoveries of other primitives of the decision problem such as acts and consequences. They construct state spaces explicitly by invoking the approach of Schmeidler and Wakker (1990) and Karni and Schmeidler (1991): a state specifies the unique consequence that is associated with every act. In particular, $S \equiv C^F$ where F is the set of feasible acts. Given this construction of the state space, a discovery leads to either a refined or genuine expansion. Specifically, discoveries of new acts lead to refined expansions of the original state space, while discoveries of new consequences lead to genuine expansions.

To illustrate their approach, consider the patient example discussed in the introduction. Let $C = \{c_1, c_2\}$ be the set of consequences, where c_1 and c_2 correspond to success and failure, respectively. Let $F = \{f_1, f_2\}$ be the set of feasible acts where f_1 and f_2 correspond to treatments A and B, respectively. Then there are four states since $S = C^F$ (see Table 1).[9] In state s_1 both

$F \setminus S$	s_1	s_2	s_3	s_4
f_1	c_1	c_1	c_2	c_2
f_2	c_1	c_2	c_1	c_2

Table 1: Original state space

treatments f_1 and f_2 are successful. In state s_2, f_1 is successful but f_2 fails, and so on.

Discoveries of new acts (Refined Expansion). Suppose a new act $\bar{f} \notin F$ is discovered. Then $S_1 \equiv C_1^{F_1}$ where $F_1 = F \cup \{\bar{f}\}$ and $C_1 = C$. Hence, in the KV-approach discoveries of new acts lead to refined expansions. Suppose that the patient discovers $\bar{f}$ (the new treatment N). The original set of acts $F = \{f_1, f_2\}$ expands to $F_1 \equiv \{f_1, f_2, \bar{f}\}$ and the expanded state space, $S_1 = C^{F_1}$, consists of eight (i.e., 2^3) states:

$F_{\bar{f}} \setminus S_{\bar{f}}$	s_1^1	s_2^1	s_3^1	s_4^1	s_5^1	s_6^1	s_7^1	s_8^1
f_1	c_1	c_1	c_2	c_2	c_1	c_1	c_2	c_2
f_2	c_1	c_2	c_1	c_2	c_1	c_2	c_1	c_2
$\bar{f}$	c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_2

Table 2: Expanded state space: new act $\bar{f}$

[8]Within the lattice structure of unawareness, Schipper (2013) characterizes awareness-dependent SEU preferences, and shows that (revealed) unawareness has a different behavioral meaning than the notion of null (i.e., impossible) events. In particular, a DM is unaware of an event if and only if the event and its complement are null events.

[9]Note that in our framework S and S_1 do no have to satisfy $S = C^F$ and $S_1 = {C_1}^{F_1}$.

Each state determines whether each of the treatments, f_1, f_2 and $\bar{f}$, is successful or not.

Notice that each original state s in S corresponds to an event E_s in S_1 that refines s; i.e., $E_s = \{s^1 \in S_1 : \exists c \in C \text{ s.t. } s^1 = (s, c)\}$. For example, state $s_1 = (c_1, c_1)$ corresponds to the new event $E_{s_1} = \{s_1^1, s_5^1\}$ that refines s_1 by incorporating consequences c_1 and c_2 associated with the new act $\bar{f}$. In other words, state s_1, originally indicating that treatments f_1 and f_2 are successful, expands now to two states, $s_1^1 = (c_1, c_1, c_1)$ and $s_5^1 = (c_1, c_1, c_2)$. In state s_1^1, all three treatments are successful while in state s_5^1, treatments f_1 and f_2 are successful while the new one $\bar{f}$ is not. Indeed, the collection of events $\{E_s\}_{s\in S}$ forms a partition of the expanded state space S_1; i.e., $S_1 = S_1^R = \bigcup_{s\in S} E_s$.

Discoveries of new consequences (Genuine Expansion). Suppose now a new consequence $\bar{c} \notin C$ is discovered. Then $S_1 = C_1^{F_1}$ where $C_1 = C \cup \{\bar{c}\}$ and $F_1 = F_{\bar{c}}$ is the set of feasible acts with extended range due to the discovery of $\bar{c}$. Since $|F| = |F_{\bar{c}}|$, discoveries of new consequences lead to genuine expansions. For example, suppose that the patient discovers that a health complication $\bar{c}$ is possible. Since $C_1 \equiv \{c_1, c_2, \bar{c}\}$, the new state space $S_1 \equiv C_1^{F_{\bar{c}}}$ consists of nine (i.e., 3^2) states:

$F_{\bar{c}} \setminus S_{\bar{c}}$	s_1^1	s_2^1	s_3^1	s_4^1	s_5^1	s_6^1	s_7^1	s_8^1	s_9^1
f_1	c_1	c_1	c_2	c_2	c_1	$\bar{c}$	c_2	$\bar{c}$	$\bar{c}$
f_2	c_1	c_2	c_1	c_2	$\bar{c}$	c_1	$\bar{c}$	c_2	$\bar{c}$

Table 3: Expanded state space: new consequence $\bar{c}$

The patient becomes aware of new states in $S_1^N = \{s_5^1, \ldots, s_9^1\}$, which is the set of conceivable states in which the health complication can happen. For example, in states s_6^1, s_8^1 and s_9^1 , f_1 (treatment A) leads to the health complication.

When the new consequence $\bar{c}$ is discovered, the original state space S genuinely expands; i.e., $|S_1^R| = |S|$. While states in $S_1^R \equiv \{s_1^1, s_2^1, s_3^1, s_4^1\}$ correspond to the original states, the states in $S_1^N \equiv S_1 \setminus S_1^R$ are new. In other words, E_s is the original state s itself; i.e., $E_s = \{s\} = \{s^1\}$ for some $s^1 \in S_1$. For instance, $E_{s_1} = \{s_1\} = \{s_1^1\} = \{(c_1, c_1)\}$.

For two acts $f, g \in \hat{F}$ and any event $E \subseteq S$, denote by $f_{-E}\, g$ the (composite) act in $\hat{F}$ that returns $g(s)$ in state $s \in E$ and $f(s')$ in state $s' \in S \setminus E$. A state $s \in S$ is said to be *null* if $f_{-s}\, p \sim_{\hat{F}} f_{-s}\, q$ for all $p, q \in \Delta(C)$, otherwise s is *nonnull*. For simplicity, we will assume that all states in S and S_1 are nonnull.[10]

3 Preferences and Consistency Notions

In this section, we illustrate our main idea of how new discoveries affect the DM's preferences. We denote by $\mathcal{F}$ a family of sets of acts corresponding to increasing levels of awareness. As reference to the original decision problem $\mathcal{D} = \{S, C, \hat{F}\}$, we fix $\hat{F} \in \mathcal{F}$ and call $\hat{F}$ the *initial set of acts.* It is the set of acts before any discovery is made. The DM's *initial preference relation* on $\hat{F}$ is denoted

[10]See Karni et al. (2020) for issues involving null states.

by $\succcurlyeq_{\hat{F}}$. When a new discovery is made, the original decision problem expands and the preference relation $\succcurlyeq_{\hat{F}}$ has to be extended to a larger domain. We denote by $\succcurlyeq_{\hat{F}_1}$ the *extended preference relation* on a set of acts $\hat{F}_1$ in an extended decision problem $\mathcal{D}_1 \equiv \{S_1, C_1, \hat{F}_1\}$.

As awareness grows, the representation of the DM's preferences might change fundamentally. To capture the idea of changing preferences formally, we assume that the initial preference is of the classical subjective expected utility form of Savage (1954).[11]

Definition 1 (Initial Preference). The initial preference relation $\succcurlyeq_{\hat{F}}$ on $\hat{F}$ is said to admit a *subjective expected utility* (SEU) representation if there exist a (unique) probability measure $\mu \in \Delta(S)$ and an expected utility functional $U : \Delta(C) \to \mathbb{R}$ such that for any $f \in \hat{F}$,

$$V^{SEU}(f) = \sum_{s \in S} U(f(s))\mu(s). \tag{2}$$

However, a DM who is initially a SEU maximizer might perceive ambiguity about newly discovered states. Hence, the extended preference relation admits the maxmin expected utility (MEU) representation of Gilboa and Schmeidler (1989).

Definition 2 (Extended Preference). The extended preference $\succcurlyeq_{\hat{F}_1}$ on $\hat{F}_1$ is said to admit a *maxmin expected utility* (MEU) representation if there exist a nonempty, convex, and compact set of probability measures $\Pi_1 \subseteq \Delta(S_1)$ and an expected utility functional $U_1 : \Delta(C_1) \to \mathbb{R}$ such that for any $f \in \hat{F}_1$,

$$V^{MEU}(f) = \min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U_1(f(s^1))\pi(s^1). \tag{3}$$

After a new discovery, the DM may not be able to form a single probability measure. Hence, her beliefs over the expanded state space are represented by a set of priors. A DM whose preferences are governed by the MEU functional is said to be *ambiguity averse*.[12]

Our goal is twofold. First, we want to behaviorally underpin the representations (2) and (3). Second, we will connect the initial preference $\succcurlyeq_{\hat{F}}$ and the extended preference $\succcurlyeq_{\hat{F}_1}$ via axioms characterizing how the DM's beliefs and tastes evolve as awareness grows.

Notice that both preference relations $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_1}$ are fully characterized by tuples (μ, U) and (Π_1, U_1) from the respective representations (2) and (3). Therefore, to link the initial and extended preferences, we will relate (μ, U) and (Π_1, U_1). In order to make sharp conclusions about how the initial preferences evolve in response to growing awareness, we will impose two consistency conditions between μ and Π_1.

Our first consistency notion requires that the extended MEU preference inherits the unambiguity of the initial SEU preference. Recall that $\{E_s\}_{s \in S}$ is the family of events in the extended state space S_1 where each E_s corresponds to an original state s in S. The first consistency notion,

[11] In Section 5, we relax this assumption and allow $\succsim_{\hat{F}}$ to be a MEU preference.

[12] In contrast, when the min-operator is replaced by a max-operator, a DM is said to be *ambiguity loving*. The results of this paper also hold when the DM is ambiguity loving.

called *Unambiguity Consistency*, requires that each event E_s is revealed to be unambiguous by the extended preference relation $\succcurlyeq_{\hat{F}_1}$. Following Nehring (1999), Ghirardato et al. (2004), and Amarante and Filiz (2007) an event E_s is *unambiguous* if each probability measure in Π_1 assigns the same value to E_s. This notion is formalized as follows.

Definition 3 (**Unambiguity Consistency**)**.** Let $\succcurlyeq_{\hat{F}}$ be a SEU preference relation on $\hat{F}$ with (μ, U) and $\succcurlyeq_{\hat{F}_1}$ be a MEU preference relation on $\hat{F}_1$ with (Π_1, U_1). Then, $\succcurlyeq_{\hat{F}_1}$ is said to be an **unambiguity consistent extension** of $\succcurlyeq_{\hat{F}}$ to $\hat{F}_1$, if each event E_s that corresponds to an original state is unambiguous according to $\succcurlyeq_{\hat{F}_1}$, i.e., for all $\pi, \pi' \in \Pi_1$ and $s \in S$,

$$\pi(E_s) = \pi'(E_s). \tag{4}$$

Unambiguity Consistency implies that old states are unambiguous. As discussed in the introduction, Unambiguity Consistency provides a novel view of ambiguity. That is, ambiguity appears since the DM treats new and old states differently.

Under Unambiguity Consistency, the DM's old belief μ on S and her new beliefs Π_1 on $\{E_s\}_{s\in S}$ can be unrelated. Likewise, the DM's risk preferences, U and U_1, may change as awareness grows. To this end, our second consistency notion, called *Likelihood Consistency*, directly connects μ and Π_1 by requiring that the DM's new beliefs Π_1 preserve the relative likelihoods of the old belief μ. This consistency notion is formalized below.

Definition 4 (**Likelihood Consistency**)**.** Let $\succcurlyeq_{\hat{F}}$ be a SEU preference relation on $\hat{F}$ with (μ, U) and $\succcurlyeq_{\hat{F}_1}$ be a MEU preference relation on $\hat{F}_1$ with (Π_1, U_1). Then, $\succcurlyeq_{\hat{F}_1}$ is said to be a **likelihood consistent extension** of $\succcurlyeq_{\hat{F}}$ to $\hat{F}_1$, if the new beliefs in Π_1 preserve the relative likelihoods of μ on S; i.e., for all $s, s' \in S$ and $\pi \in \Pi_1$:

$$\frac{\mu(s)}{\mu(s')} = \frac{\pi\big(E_s\big)}{\pi\big(E_{s'}\big)}. \tag{5}$$

There is one important remark. Likelihood Consistency implies that the rankings over old acts are preserved. In the context of refined expansions of the original state space, Likelihood Consistency implies Unambiguity Consistency since $\sum_{s\in S} \pi\big(E_s\big) = 1$. However, the two consistency notions are independent in the case of genuine expansions since $\sum_{s\in S} \pi\big(E_s\big) < 1$.[13]

3.1 Illustrations and Behavioral Implications

To illustrate our consistency notions, consider again the patient example. When confronted with the original model (see Table 1), suppose the patient believes that each state is equally likely. That is, her belief μ on S is given by $\mu(s_1) = \mu(s_2) = \mu(s_3) = \mu(s_4) = \frac{1}{4}$. Thus the patient with a SEU preference $\succcurlyeq_{\hat{F}}$ is indifferent between the two treatments (f_1 and f_2) and any mixture thereof (i.e., $\alpha f_1 + (1 - \alpha) f_2 \sim_{\hat{F}} f_1 \sim_{\hat{F}} f_2$).

[13] When initial and extended preferences are SEU, Likelihood Consistency coincides with the reverse Bayesian updating of KV. However, Likelihood Consistency is more general when ambiguity is allowed.

Consider now the patient's extended preference $\succsim_{\hat{F}_1}$ after the new treatment $\bar{f}$ is discovered. In this case, each original state $s \in S$ admits a finer description depending on whether her treatment $\bar{f}$ is successful or not. Hence, S_1 has 8 states and is given by Table 2.

The patient might not be able to "split" her initial belief $\mu(s_i)$ across the new states s_i^1 and s_{i+4}^1 with $i = 1, \ldots, 4$. For example, she might consider the following set of priors:

$$(6) \qquad \Pi_1 = \begin{cases} \pi \in \Delta(S_1) : & \pi(s_i^1) + \pi(s_{i+4}^1) = \pi(E_{s_i}) = \frac{\beta}{2} \qquad \text{for } i = 1, 2; \\ & \pi(s_i^1) + \pi(s_{i+4}^1) = \pi(E_{s_i}) = (1-\beta)\frac{1}{2} \quad \text{for } \; i = 3, 4, \end{cases}$$

where $\beta \in [0, 1]$. Notice that each event $E_{s_i} = \{s_i^1, s_{i+4}^1\}$ in S_1 that corresponds to the original state $s_i \in S$ is unambiguous while each newly discovered state $s_i^1 \in S^1$ is ambiguous. Thus, for any $\beta \in [0, 1]$, her extended preference $\succsim_{\hat{F}_1}$ preserves unambiguity of the initial preference $\succsim_{\hat{F}}$.

When $\beta = \frac{1}{2}$, the patient's extended MEU preference is likelihood consistent since the set of priors maintains the relative likelihoods of μ as $\mu(s_i)/\mu(s_j) = \frac{1}{4}/\frac{1}{4} = \pi(E_{s_i})/\pi(E_{s_j}) = \frac{1}{4}/\frac{1}{4}$. As remarked before, Likelihood Consistency implies Unambiguity Consistency under refined expansions. Moreover, the ambiguity averse patient is still indifferent between f_1, f_2, and any of their mixtures. However, she strictly prefers either of the standard treatments to the new treatment $\bar{f}$ (i.e., $\alpha f_1 + (1-\alpha) f_2 \sim_{\hat{F}_1} f_1 \sim_{\hat{F}_1} f_2 \succ_{\hat{F}_1} \bar{f}$).

Consider now the case in which the new consequence $\bar{c}$ is discovered (see Table 3). Since the states s_5^1 through s_9^1 are newly discovered, the patient might not be able to form a single prior over S_1. Instead, she might consider a set of priors Π_1. For instance, consider the following set of priors:

$$(7) \qquad \Pi_1 = \left\{\pi \in \Delta(S_1) : \pi(s_i^1) = \frac{\gamma}{16} \text{ and } \gamma \in [1, \overline{\gamma}] \;\; \text{for all } i = 1, \ldots, 4\right\}.$$

Under genuine expansions, Likelihood Consistency and Unambiguity Consistency are independent. For example, when $\overline{\gamma} = 1$, the extended preference with respect to Π_1 reveals that the original states s_1^1 through s_4^1 are unambiguous while the newly discovered states $s_5^1, s_6^1, s_7^1, s_8^1$ and s_9^1 are ambiguous. However, when $\overline{\gamma} = 2$, the original states s_1^1, s_2^1, s_3^1, and s_4^1 are ambiguous and Unambiguity Consistency is violated. Nevertheless, Π_1 still preserves the relative likelihoods of μ since $\mu(s_i)/\mu(s_j) = \frac{1}{4}/\frac{1}{4} = \pi(E_{s_i})/\pi(E_{s_j}) = \frac{\gamma}{16}/\frac{\gamma}{16}$.[14] Moreover, the patient's preference has different behavioral implications compared to the case where a new act is discovered. Specifically, the patient, who is still indifferent between old treatments f_1 and f_2, strictly prefers any mixture of them over each of f_1 and f_2 alone. In other words, the patient reveals ambiguity aversion in the standard sense (i.e., $\alpha f_1 + (1-\alpha) f_2 \succ_{\hat{F}_1} f_1 \sim_{\hat{F}_1} f_2$).

To sum up, ambiguity arises differently depending on whether refined or genuine expansion has occurred. Under refined expansion, only new acts are ambiguous, while under genuine expansion, old acts might be ambiguous. Hence, two different types of discoveries can be behaviorally disentangled. However, this distinction is difficult when both the initial and extended preferences are

[14] Similar to (6), it is not difficult to construct an example in which s_1^1, s_2^1, s_3^1, and s_4^1 are unambiguous, but Π_1 does not preserve the relative likelihoods of μ.

SEU. Moreover, regardless of what is discovered, (i) Unambiguity Consistency implies that new states may be ambiguous while old states are unambiguous, and (ii) Likelihood Consistency implies that the rankings over old acts are preserved (i.e., $f_1 \sim_{\hat{F}} f_2$ and $f_1 \sim_{\hat{F}_1} f_2$).

4 Behavioral Foundations

In this section, we axiomatically characterize Unambiguity and Likelihood Consistency. Although we highlighted the behavioral differences between refined and genuine expansions, our results are unified in a way that axioms and characterization theorems are the same in both contexts.

4.1 Basic Preference Structure

We have the initial preference $\succsim_{\hat{F}}$ from the initial decision problem $\mathcal{D} = (S, C, \hat{F})$ and the extended preference $\succsim_{\hat{F}_1}$ from the new decision problem $\mathcal{D}_1 = (S_1, C_1, \hat{F}_1)$.

For all $f, g \in \hat{F}$, and $\alpha \in [0,1]$, $\alpha f + (1-\alpha) g \in \hat{F}$ is the act $h \in \hat{F}$ defined by $h(s) = \alpha f(s) + (1-\alpha) g(s)$ for any $s \in S$. Then, $\hat{F}$ is a convex subset of a linear space. First, we assume that both $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_1}$ satisfy the following basic axioms:

(A.1) (**Weak order**) For all $\hat{F} \in \mathcal{F}$, the preference relation $\succsim_{\hat{F}}$ is transitive and complete.

(A.2) (**Archimedean**) For all $\hat{F} \in \mathcal{F}$ and $f, g, h \in \hat{F}$, if $f \succ_{\hat{F}} g$ and $g \succ_{\hat{F}} h$, then there exist $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha) h \succ_{\hat{F}} g$ and $g \succ_{\hat{F}} \beta f + (1-\beta) h$.

(A.3) (**Monotonicity**) For all $\hat{F} \in \mathcal{F}$ and $f, g \in \hat{F}$, if $f(s) \succsim_{\hat{F}} g(s)$ for all $s \in S$, then $f \succsim_{\hat{F}} g$.

(A.4) (**Nondegeneracy**) For all $\hat{F} \in \mathcal{F}$, there are $f, g \in \hat{F}$ such that $f \succ_{\hat{F}} g$.

To capture our idea that the DM's behavior might change fundamentally as awareness grows, we allow for $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_1}$ to belong to different families of preferences. In particular, we assume that the initial preference relation $\succsim_{\hat{F}}$ satisfies the Independence Axiom:

(A.5) (**Independence**) For all $f, g, h \in \hat{F}$, and $\alpha \in (0,1]$, $f \succsim_{\hat{F}} g$ if and only if $\alpha f + (1-\alpha) h \succsim_{\hat{F}} \alpha g + (1-\alpha) h$.

That is, the initial preference relation $\succsim_{\hat{F}}$ will admit the SEU representation (1) with respect to a unique probability distribution μ on S and an expected utility functional $U : \Delta(C) \to \mathbb{R}$ (e.g., see Anscombe and Aumann, 1963). However, the extended preference $\succsim_{\hat{F}_1}$ might violate the Independence Axiom allowing for ambiguity.

4.2 MEU and Unambiguity Consistency

In this subsection, we obtain the MEU representation of the extended preference $\succsim_{\hat{F}_1}$ that is an unambiguity consistent extension of the initial SEU preference $\succsim_{\hat{F}}$.

We introduce an axiom, called *Negative Unambiguity Independence* (henceforth, NUI). Roughly speaking, NUI specifies how the new and old acts are evaluated by the extended preference $\succcurlyeq_{\hat{F}_1}$. The axiom contains two parts. The first part states that if a new act f is weakly preferred to a lottery q, then mixing the act with another act g is at least as good as mixing the lottery with g. The second part directly connects the new acts with the old, binary acts (called bets). Specifically, it requires that bets on the events that correspond to original states cannot be used to hedge against ambiguity of the new acts.

Recall that, for each initial state $s \in S$, E_s denotes the event in S_1 which corresponds to $s \in S$. We can now state NUI formally.

(A.6) (**Negative Unambiguity Independence (NUI)**) For all $f, g \in \hat{F}_1$, $q \in \Delta(C_1)$ and $\alpha \in [0,1]$,

$$\text{if } \; f \succcurlyeq_{\hat{F}_1} q, \;\; \text{then } \; \alpha f + (1-\alpha)g \succcurlyeq_{\hat{F}_1} \alpha q + (1-\alpha)g,$$

and when $g = p_{E_s} r$ for some $s \in S$ and $p, r \in \Delta(C_1)$,

$$f \sim_{\hat{F}_1} q \;\text{ if and only if }\; \alpha f + (1-\alpha)g \sim_{\hat{F}_1} \alpha q + (1-\alpha)g.$$

The first part suggests that the objective lottery q, which is a constant act and thus ambiguity free, suffers more (or gains less) than the subjective act f from mixtures that eliminate its objective appeal. However, the second part suggests that the objective lottery q and the subjective act f equally suffer (or equally gain) when they are mixed with the previously known bets. Note that NUI does not explicitly depend on the initial preference since $\succsim_{\hat{F}}$ is a SEU preference. However, it implicitly depends on the initial preference since E_s is the event that corresponds to the original, unambiguous state.

The spirit of our axiom is reminiscent of Negative Certainty Independence axiom introduced by Dillenberger (2010) and used by Cerreia-Vioglio et al. (2015) to characterize the Cautious Expected Utility theory in the context of choice under risk.[15] However, in our setup, NUI has different behavioral implications since we allow for ambiguity.

In the context of growing awareness, NUI has two behavioral consequences. First, the axiom guarantees that the extended preference relation admits a MEU representation. Second, NUI implies that all the events that correspond to the original states are unambiguous.

We can now formalize our main representation theorem.

Theorem 1. *The following statements are equivalent:*

(i) $\succcurlyeq_{\hat{F}}$ *satisfies axioms (A.1)-(A.5) and* $\succcurlyeq_{\hat{F}_1}$ *satisfies axioms (A.1)-(A.4) and NUI.*

[15] Riella (2015) also uses the Negative Certainty Independence axiom in the context of choice under uncertainty. In particular, he extends the main result of Cerreia-Vioglio et al. (2015) and obtains a single-prior expected multiple-utility representation for incomplete preferences. The idea behind NUI is also similar to the Caution axiom of Gilboa et al. (2010). In their setup, the Caution axiom connects two preferences, called objective and subjective rationality relations. The former admits a MEU representation while the latter admits a representation à la Bewley (2002).

(ii) There exist non-constant and affine functions $U : \Delta(C) \to \mathbb{R}$ and $U_1 : \Delta(C_1) \to \mathbb{R}$, a probability measure $\mu \in \Delta(S)$, and a nonempty, convex and compact set of probability measures $\Pi_1 \subseteq \Delta(S_1)$ such that $\succcurlyeq_{\hat{F}}$ is a SEU preference with (μ, U), $\succcurlyeq_{\hat{F}_1}$ is a MEU preference with (Π_1, U_1), and $\succcurlyeq_{\hat{F}_1}$ is an unambiguity consistent extension of $\succcurlyeq_{\hat{F}}$. Moreover, U and U_1 are unique up to a positive linear transformation and μ and Π_1 are unique.

Theorem 1 characterizes the SEU and MEU representations of $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_1}$ and simultaneously establishes that the extended preference $\succcurlyeq_{\hat{F}_1}$ preserves unambiguity of $\succcurlyeq_{\hat{F}}$ as awareness grows.

Remark 1. Unambiguity Consistency has a stronger behavioral implication in the context of genuine expansion. In this case, the event $S_1^R = \cup_{s \in S} E_s$ in S_1 corresponds to the original state space S; i.e., $|S_1^R| = |S|$. Hence, the extended MEU preference admits an additive decomposition across the unambiguous event S_1^R and its complement S_1^N. This observation is formally stated in the following corollary.

Corollary 1. (Genuine expansion) Suppose $|S_1^R| = |S|$. Let $\succcurlyeq_{\hat{F}}$ be a SEU preference and $\succcurlyeq_{\hat{F}_1}$ be a MEU preference with (Π_1, U_1) as in Theorem 1. Then, there are $\delta \in (0, 1)$, $\tilde{\mu} \in \Delta(S)$, and a nonempty, convex and compact set $\tilde{\Pi} \subset \Delta(S_1^N)$ such that for any $f \in \hat{F}_1$,

$$\min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U_1\big(f(s^1)\big)\pi(s^1) = \delta\Big(\sum_{s \in S} \tilde{\mu}(s) U_1\big(f(E_s)\big)\Big) + (1-\delta)\Big(\min_{\tilde{\pi} \in \tilde{\Pi}} \sum_{s^1 \in S_1^N} U_1\big(f(s^1)\big)\tilde{\pi}(s^1)\Big).$$

Under Unambiguity Consistency, the DM who discovers a new consequence shifts $(1-\delta)$ of the original probability mass to the set of newly discovered states, S_1^N. In other words, $(1-\delta)$ is the subjective probability that one of the newly discovered states will occur. However, the DM might not know how to "split" the probability mass $(1-\delta)$ across the newly discovered states in S_1^N and thus she may perceive the new states as ambiguous. However, in general Unambiguity Consistency does not imply that $\succcurlyeq_{\hat{F}_1}$ is additively separable across the unambiguous events $\{E_s\}_{s \in S}$.

Notice that under Unambiguity Consistency, the DM solely inherits unambiguity property of her original beliefs in response to growing awareness. The old and new beliefs might be unrelated. To link μ and Π_1 or U and U_1, additional axioms are required.

To ensure that risk attitudes are not affected by awareness (i.e., $U = U_1$ on $\Delta(C)$), we impose an axiom called *Invariant Risk Preferences.*[16] It requires that the DM's rankings of lotteries remain the same at any awareness level. Formally,

(A.7) (**Invariant Risk Preferences**) For all $p, q \in \Delta(C)$, $p \succcurlyeq_{\hat{F}} q$ if and only if $p \succcurlyeq_{\hat{F}_1} q$.

By requiring that $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_1}$ jointly satisfy (A.7), we get the following lemma.

Lemma 1. *Let $\succcurlyeq_{\hat{F}}$ be a SEU preference with (μ, U) and $\succcurlyeq_{\hat{F}_1}$ be a MEU preference with (Π_1, U_1). If $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_1}$ jointly satisfy Invariant Risk Preferences, then there are $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $U(p) = \alpha U_1(p) + \beta$ for any $p \in \Delta(C)$.*

[16]The axiom was introduced by KV in their axiomatization of reverse Bayesianism.

As a consequence, when $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_1}$ jointly satisfy Invariant Risk Preferences, in addition to the axioms of Theorem 1, then the extended MEU preference $\succcurlyeq_{\hat{F}_1}$ preserves both unambiguity and the risk attitude of the initial SEU preference $\succcurlyeq_{\hat{F}}$.

4.3 MEU and Likelihood Consistency

In this subsection, we behaviorally characterize Likelihood Consistency. To this end, we impose another axiom called *Binary Awareness Consistency* (BAC) in addition to a weak version of NUI. Recall that, for any $f, g \in \hat{F}$ and $T \subseteq S$, $f_{-T}\, g$ is the act in $\hat{F}$ that returns $g(s)$ in state $s \in T$ and $f(s')$ in state $s' \in S \setminus T$. The BAC axioms directly connects the initial preference $\succcurlyeq_{\hat{F}}$ and the extended preference $\succcurlyeq_{\hat{F}_1}$ in the following way.

(A.8) (**Binary Awareness Consistency (BAC)**) For all $p, p', q, q', r \in \Delta(C)$ and all $s \in S$,

$$(p_{-s}\, q) \succcurlyeq_{\hat{F}} (p'_{-s}\, q') \text{ if and only if } \big(r_{-S_1^R}(p_{-E_s}\, q)\big) \succcurlyeq_{\hat{F}_1} \big(r_{-S_1^R}(p'_{-E_s}\, q')\big).$$

Roughly speaking, BAC requires that rankings over the old binary acts $(p_{-s}\, q)$ and $(p'_{-s}\, q')$ are not affected by growing awareness. Since $S_1^R = S_1$ in the context of refined expansions, $\big(r_{-S_1^R}(p_{-E_s}\, q)\big)$ is a "projection" of the old act $(p_{-s}\, q)$ on $\hat{F}_1$. In the context of genuine expansions, BAC is also reminiscent of the Sure-Thing Principle constrained to binary acts.[17] By imposing BAC, we can weaken NUI in the following way.

(A.9) (**Weak Negative Unambiguity Independence (WNUI)**) For all $f, g \in \hat{F}_1$, $q \in \Delta(C_1)$ and $\alpha \in [0, 1]$,

$$\text{if } f \succcurlyeq_{\hat{F}_1} q, \text{ then } \alpha f + (1-\alpha) g \succcurlyeq_{\hat{F}_1} \alpha q + (1-\alpha) g,$$

and when $g = p$ for some $p \in \Delta(C_1)$,

$$f \sim_{\hat{F}_1} q \text{ if and only if } \alpha f + (1-\alpha) g \sim_{\hat{F}_1} \alpha q + (1-\alpha) g.$$

The first part of WNUI is identical to the first part of NUI. Hence, the first part suggests that the objective lottery q suffers more (or gains less) than the subjective act f from mixtures that eliminate its objective appeal. However, the second part of WNUI requires that a subjective act f and an objective lottery q equally suffer (or equally gain) when they are mixed with objective lotteries.[18]

Our second representation result is stated below.

Theorem 2. *The following statements are equivalent:*

(i) $\succcurlyeq_{\hat{F}}$ *satisfies axioms (A.1)-(A.5),* $\succcurlyeq_{\hat{F}_1}$ *satisfies axioms (A.1)-(A.4) and WNUI, and* $\succcurlyeq_{\hat{F}}$ *and* $\succcurlyeq_{\hat{F}_1}$ *jointly satisfy BAC.*

[17] Moreover, BAC can be thought as a weak version of dynamic consistency or Bayesian updating.

[18] Note that the second part of WNUI is obtained from the second part of NUI by setting $p = r$ when $g = p_{E_s} r$.

(ii) There exist a non-constant and affine function $U : \Delta(C_1) \to \mathbb{R}$, a probability measure $\mu \in \Delta(S)$, and a nonempty, convex and compact set of probability measures $\Pi_1 \subseteq \Delta(S_1)$ such that $\succsim_{\hat{F}}$ is a SEU preference with (μ, U), $\succsim_{\hat{F}_1}$ is a MEU preference with (Π_1, U), and $\succsim_{\hat{F}_1}$ is a likelihood consistent extension of $\succsim_{\hat{F}}$. Moreover, U is unique up to a positive linear transformation and μ and Π_1 are unique.

Theorem 2 characterizes the representations (1) and (2) with $U = U_1$ on $\Delta(C)$, and simultaneously establishes that $\succsim_{\hat{F}_1}$ is a likelihood consistent extension of $\succsim_{\hat{F}}$.

Remark 2. Theorem 2 shows that BAC implies Invariant Risk Preferences. This is because BAC requires that the rankings over old acts including constant acts are preserved by the extended preference.[19] Moreover, if all the newly discovered states are unambiguous, Theorem 2 provides an alternative foundation of reverse Bayesianism of Karni and Vierø (2013, Theorems 1-2).[20]

Interestingly, depending on the type of state space expansion, Theorem 2 might have different implications about Unambiguity Consistency. For example, under refined expansions Theorem 2 also characterizes Unambiguity Consistency.

Corollary 2. (Refined expansion) Suppose that $S_1 = S_1^R$. Let $\succsim_{\hat{F}}$ be a SEU preference with (μ, U) and $\succsim_{\hat{F}_1}$ be a MEU preference with (Π_1, U) as in Theorem 2. Then $\succsim_{\hat{F}_1}$ is an unambiguity and likelihood consistent extension of $\succsim_{\hat{F}}$, i.e., for all $s \in S$ and all $\pi, \tilde{\pi} \in \Pi_1$,

$$\mu(s) = \pi\big(E_s\big) = \tilde{\pi}\big(E_s\big). \tag{8}$$

Therefore, in the context of refined expansions, the extended MEU preference inherits *all* the properties of the initial SEU preferences, including the DM's old beliefs as well as her old risk attitude. However, under genuine expansions the extended preference $\succsim_{\hat{F}_1}$ in Theorem 2 is not necessarily unambiguity consistent. The following corollary of Theorem 2 shows that events in $\{E_s\}_{s \in S}$ (even $S_1^R = \bigcup_{s \in S} E_s$) are possibly ambiguous.

Corollary 3. (Genuine expansion) Suppose that $|S_1^R| = |S|$. Let $\succsim_{\hat{F}}$ be a SEU preference with (μ, U) and $\succsim_{\hat{F}_1}$ be a MEU preference with (Π_1, U) as in Theorem 2. Then there is a set $[\underline{\delta}, \overline{\delta}] \times \tilde{\Pi} \subset [0,1] \times \Delta(S_1^N)$ such that for any $f \in \hat{F}_1$,

$$\min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U\big(f(s^1)\big)\pi(s^1) = \min_{(\delta, \tilde{\pi}) \in [\underline{\delta}, \overline{\delta}] \times \tilde{\Pi}} \Big\{ \delta \sum_{s \in S} \mu(s) U\big(f(s)\big) + (1-\delta) \sum_{s^1 \in S_1^N} U\big(f(s^1)\big)\tilde{\pi}(s^1) \Big\}.^{21}$$

[19] A similar result is obtained by Dominiak and Tserenjigmid (2018) for SEU preferences in the context of reverse Bayesianism of Karni and Vierø (2013).

[20] Karni and Vierø (2013) require two axioms in addition to SEU axioms; Invariant Risk Preferences and Projection Consistency in the context of discovering acts and Awareness Consistency in the context of discovering consequences. Our BAC is weaker than both Awareness Consistency and Projection Consistency.

[21] Technically, $\underline{\delta} = \overline{\delta}$ is allowed and in this case, we obtain the representation in Corollary 1. However, when $\underline{\delta} \neq \overline{\delta}$ Unambiguity Consistency is violated.

We conclude this section by summarizing our characterization results in Table 4. Unambiguity Consistency is characterized in Theorem 1 by NUI. Likelihood Consistency is characterized in Theorem 2 by WNUI and BAC. Since Likelihood Consistency implies Unambiguity Consistency under refined expansions, Theorem 2 also characterizes both consistency notions. Finally, Theorems 1 and 2 characterize Unambiguity and Likelihood Consistency by NUI and BAC in the context of genuine expansions.

	refined expansion	genuine expansion
Unambiguity Consistency	Theorem 1 (NUI)	Theorem 1 (NUI)
Likelihood Consistency	Theorem 2 (WNUI+BAC)	Theorem 2 (WNUI+BAC)
Unambiguity and Likelihood	Theorem 2 (WNUI+BAC)	Theorems 1, 2 (NUI+BAC)

Table 4: Summary of characterization results

4.4 Dynamic Consistency and Rectangularity

In this section, we discuss the relationships between dynamic consistency and our consistency notions. For SEU preferences, it is known that dynamic consistency is equivalent to Bayesian updating (see Epstein and Breton, 1993; Ghirardato, 2002). However, for MEU preferences, dynamic consistency is equivalent to *rectangularity* of the set of priors (see Epstein and Schneider, 2003).

Let us first define dynamic consistency and rectangularity in our context. Let $\mathcal{P}$ be a partition of S and $\mathcal{P}_1$ be the partition of S_1 such that $\mathcal{P}_1 \equiv \{E(P)\}_{P\in\mathcal{P}} \cup \{S_1^N\}$ where $E(P) = \bigcup_{s\in P} E_s$. Let $\succsim_{\hat{F}_1}$ be the unconditional MEU preference with (Π_1, U), and for each $P_1 \in \mathcal{P}_1$, let $\succsim_{\hat{F}_1}^{P_1}$ be the conditional MEU preference with (Π_{P_1}, U).[22] Then we say that the MEU preferences $\succsim_{\hat{F}_1}$ and $\{\succsim_{\hat{F}_1}^{P_1}\}_{P_1\in\mathcal{P}_1}$ satisfy *dynamic consistency* if for any $P_1 \in \mathcal{P}_1$ and $f, g, h \in \hat{F}_1$, $f_{P_1}h \succsim_{\hat{F}_1} g_{P_1}h$ if and only if $f \succsim_{\hat{F}_1}^{P_1} g$. By the result of Epstein and Schneider (2003), dynamic consistency is satisfied if and only if Π_1 is $\mathcal{P}_1$*-rectangular*; i.e., Π_1 satisfies the following recursivity property:

$$\Pi_1 = \Big\{ \sum_{P_1\in\mathcal{P}_1} \pi_1'(P_1)\cdot\pi_{P_1} \mid \pi_1' \in \Pi_1 \text{ and } \pi_{P_1'} \in \Pi_{P_1'} \text{ for each } P_1' \in \mathcal{P}_1 \Big\}. \tag{9}$$

In this section, since the initial preference relation is SEU, we assume that $\mathcal{P} = \{\{s\}\}_{s\in S}$ and $\mathcal{P}_1 = \{E_s\}_{s\in S} \cup \{S_1^N\}$. It turns out that the relationship between our consistency notions and rectangularity varies with the type of state space expansion. In the context of refined expansion, the joint assumption of Unambiguity and Likelihood Consistency is not sufficient for Π_1 to be $\mathcal{P}_1$-rectangular. The reason is that rectangularity requires the unconditional MEU preference to be additively separable across the events in $\mathcal{P}_1$ in the case of refined expansion as formalized below. However, additive-separability is not guaranteed by Unambiguity and Likelihood Consistency.

[22] Π_{P_1} is the Full-Bayesian update of Π_1; i.e., $\Pi_{P_1} = \{\pi_{P_1} \in \Delta(S_1) | \pi_{P_1}$ is the Bayesian update of some $\pi \in \Pi_1\}$.

Corollary 4 (Refined expansion)**.** Suppose $S_1 = S_1^R$. If $\succcurlyeq_{\hat{F}_1}$ is a MEU preference with (Π_1, U) and is an unambiguous (and/or likelihood) consistent extension of $\succcurlyeq_{\hat{F}}$, then Π_1 is $\mathcal{P}_1$-rectangular if and only if $\succcurlyeq_{\hat{F}_1}$ is $\mathcal{P}_1$-additive-separable; i.e.,

$$\min_{\pi \in \Pi_1} \{ \sum_{s^1 \in S_1} U(f(s^1))\pi(s^1) \} = \sum_{P_1 \in \mathcal{P}_1} \min_{\pi \in \Pi_1} \{ \sum_{s \in P_1} U(f(s^1))\pi(s^1) \}.$$

For genuine expansions, each of the two consistency notions individually implies rectangularity.

Corollary 5 (Genuine expansion)**.** Suppose $|S| = |S_1^R|$. If $\succcurlyeq_{\hat{F}_1}$ is a MEU preference with (Π_1, U) and is an unambiguous (or likelihood) consistent extension of $\succcurlyeq_{\hat{F}}$, then Π_1 is $\mathcal{P}_1$-rectangular.

In this case, Unambiguity Consistency implies additive-separability, hence, rectangularity. However, Likelihood Consistency does not imply additive-separability but yet implies rectangularity (see the representation in Corollary 3).

5 Allowing for Initial Ambiguity Aversion

So far we have focused on a situation in which the DM's preferences evolve from SEU preferences to MEU preferences after learning a new discovery. The key insight from the previous sections is to identify and characterize a new source of ambiguity. A natural next step is to ask how does the DM react to another discovery when her preference is already MEU. To this end, this section focuses on situations where both the initial and extended preferences, $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_1}$, are MEU preferences. The main objective is then to extend our two consistency notions in this more general environment and obtain characterization results that generalize Theorems 1 and 2. The new results illustrate that our framework can be extended to settings with more than two periods, since any consecutive two periods can be captured by our framework.[23]

Throughout this section, we assume that the initial preference $\succcurlyeq_{\hat{F}}$ is a MEU preference with (Π, U) and the extended preference $\succcurlyeq_{\hat{F}_1}$ is also a MEU preference with (Π_1, U_1). The initial preference relation here admits two different interpretations. In the first and main interpretation, the initial preference $\succcurlyeq_{\hat{F}}$ is a consequence of an evolution of another SEU preference (as characterized in Section 4). In this case, we study how the DM's preferences evolve after a second discovery. In the second interpretation, the initial preference relation represents the DM's behavior before any discovery is made and the DM's preference is ambiguity averse for different reasons (e.g., the DM is ambiguity averse due to unknown probabilities as in the Ellsberg experiment). In this case, we study a coexistence of two different types of ambiguity.

Our first task is to extend Unambiguity Consistency and Likelihood Consistency by connecting the initial MEU preference $\succcurlyeq_{\hat{F}}$ with (Π, U) and the extended MEU preference $\succcurlyeq_{\hat{F}_1}$ with (Π_1, U_1). Indeed, we mostly focus on connecting the set of priors Π and Π_1. We then discuss dynamic consistency and rectangularity. We start with the extension of Likelihood Consistency.

[23] Indeed, modeling awareness in the context of intertemporal choice is nontrivial and it requires a separate treatment (see Vierø (2017)). Combining such an extension with ambiguity is beyond the scope of the current paper.

5.1 Likelihood Consistency and Reverse Full-Bayesianism

In KV, both the initial and extended preferences are SEU and the initial prior is the Bayesian update of the extended prior. Hence the evolution of SEU beliefs in KV is called *reverse Bayesianism.* In this subsection, we define and characterize an evolution of MEU preferences called *reverse full-Bayesianism.* It is an extension of reverse Bayesianism to MEU preferences according to which the initial set of priors is the full-Bayesian (i.e., prior-by-prior) update of the extended set of priors.

Definition 5 (Reverse Full-Bayesianism). Let $\succcurlyeq_{\hat{F}}$ be a MEU preference with (Π, U) and $\succcurlyeq_{\hat{F}_1}$ be a MEU preference with (Π_1, U_1). Then, $\succcurlyeq_{\hat{F}_1}$ is said to be a **reverse full-Bayesian extension** of $\succcurlyeq_{\hat{F}}$ to $\hat{F}_1$ if

$$\Pi = \left\{ \pi \in \Delta(S) \middle| \exists \pi_1 \in \Pi_1 \text{ such that for any } s, s' \in S,\ \frac{\pi(s)}{\pi(s')} = \frac{\pi_1(E_s)}{\pi_1(E_{s'})} \right\}. \tag{10}$$

Under reverse full-Bayesianism, the DM extends her old beliefs Π on S to new beliefs Π_1 on S_1 in such a way that her new beliefs maintain - prior by prior - the relative likelihoods on S of her old beliefs. Hence, if the initial preference is SEU (i.e., Π is singleton), then reverse full-Bayesianism is equivalent to Likelihood Consistency. Indeed, when both the initial and extended preferences are SEU, we obtain the reverse Bayesianism of KV.

To characterize reverse full-Bayesianism, we impose two axioms that connect the initial preference $\succcurlyeq_{\hat{F}}$ and the extended preference $\succcurlyeq_{\hat{F}_1}$. The first axiom, *Certainty Equivalence Consistency* (CEC), connects the initial preference $\succcurlyeq_{\hat{F}}$ and the extended preference $\succcurlyeq_{\hat{F}_1}$ by requiring that certainty equivalences of old acts do not change after a discovery. Recall that for any $f \in \hat{F}$ and $g \in \hat{F}_1$, we denote an act in $\hat{F}_1$ that returns $f(s)$ for each state$s_1 \in E_s$ where $s \in S$ and g otherwise by $(f(s)\, E_s)_{s \in S} \cup g$. Keeping this notation in mind, we can formalize CEC.

(A.10) (**Certainty Equivalence Consistency (CEC)**) For any $f \in \hat{F}$ and $p \in \Delta(C)$,

$$f \sim_{\hat{F}} p \ \text{ implies } \ (f(s)\, E_s)_{s \in S} \cup p \sim_{\hat{F}_1} p.$$

In words, p is essentially the certainty equivalent of an old act f in $\hat{F}$. Then p is still the certainty equivalent of the old act $(f(s)\, E_s)_{s \in S} \cup p$ in $\hat{F}_1$ that returns f on S_1^R and p otherwise.

(A.11) (**Negative Certainty Equivalence Independence (NCEI)**) For any $f \in \hat{F}$ and $p, q \in \Delta(C)$, if $f(s) \succ_{\hat{F}} q$ for each $s \in S$ or $q \succ_{\hat{F}} f(s)$ for each $s \in S$, then

$$f \sim_{\hat{F}} p \ \text{ implies } \ (f(s)\, E_s)_{s \in S} \cup q \succsim_{\hat{F}_1} p\, E(S) q.$$

The idea behind NCEI is similar to NUI. Roughly speaking, NCEI suggests that the objective lottery p, which is ambiguity free, suffers more (or gains less) than the subjective act f from "composition" (or a projection that makes f and p composite acts) that eliminates its objective appeal. However, note that NCEI imposes the above property only when q dominates (or is dominated by) f.

In the context of refined extensions (i.e., $S_1 = S_1^R$), q in NCEI becomes redundant and CEC implies NCEI. However, CEC and NCEI are independent in general.[24]

We now can characterize the theory of reverse full-Bayesianism.

Theorem 3. *The initial preference relation $\succsim_{\hat{F}}$ is a MEU preference with (Π, U) and the extended preference relation $\succsim_{\hat{F}_1}$ is a MEU preference with (Π_1, U_1). The preferences jointly satisfy CEC and NCEI if and only if $\succsim_{\hat{F}_1}$ is a reverse full-Bayesian extension of $\succcurlyeq_{\hat{F}}$ and there are $\alpha, \beta \in \mathbb{R}$ with $\alpha > 0$ such that $U(p) = \alpha\, U_1(p) + \beta$ for any $p \in \Delta(C)$.*

Theorem 3 shows that CEC and NCEI characterize reverse full-Bayesiansim. Just like Theorem 2, CEC also requires the initial and extended risk preferences to be the same. Hence, Theorem 3 extends Theorem 2.

5.2 Partition-Dependent Unambiguity Consistency

The idea behind our consistency notions is that the extended preference relation inherits certain structures from the initial preference relation. Unambiguity Consistency requires that the extended MEU preferences inherit unambiguity of old states. To extend Unambiguity Consistency to this more general setting with two MEU preferences, we fix a partition $\mathcal{P}$ of S and assume that each event $P \in \mathcal{P}$ is $\succsim_{\hat{F}}$-unambiguous. This assumption allows for two interpretations. First, when $\succsim_{\hat{F}}$ is a SEU preference as in the previous section, $\mathcal{P}$ is the partition that consists of only singleton sets; i.e., $\mathcal{P} = \{\{s\}\}_{s\in S}$. Second, $\mathcal{P}$ is the partition that is obtained from a previous discovery under Unambiguity Consistency.[25] Given $\mathcal{P}$, let $\mathcal{P}_1 \equiv \{E(P)\}_{P\in\mathcal{P}} \cup \{S_1^N\}$. Then our extension of Unambiguity Consistency will require that any event in $P_1 \in \mathcal{P}_1$ is $\succsim_{\hat{F}_1}$-unambiguous. For simplicity, we introduce the following definitions.

Definition 6 ($\mathcal{E}$-unambiguity)**.** For any partition $\mathcal{E}$ of S, a MEU preference relation $\succcurlyeq$ with (Π, U) is **$\mathcal{E}$-unambiguous** if for any event $E \in \mathcal{E}$ and priors $\pi, \pi' \in \Pi$, $\pi(E) = \pi'(E)$.

Definition 7 ($\mathcal{E}$-Independence)**.** For any partition $\mathcal{E}$ of S, a preference relation $\succsim$ on $\hat{F}$ satisfies **$\mathcal{E}$-Independence** if for any $E \in \mathcal{E}$ and any $f \in \hat{F}$, $p, q, r \in \Delta(C)$, and $\alpha \in (0, 1]$, $f \sim q$ if and only if $\alpha f + (1 - \alpha)(p\,_E\, r) \sim \alpha\, q + (1 - \alpha)(p\,_E\, r)$.

We now can formally define the extension of Unambiguity Consistency.

Definition 8 (PD-Unambiguity Consistency). Suppose the initial preference relation $\succcurlyeq_{\hat{F}}$ is a $\mathcal{P}$-unambiguous MEU preference and the extended preference relation $\succcurlyeq_{\hat{F}_1}$ is a MEU preference with (Π_1, U_1). Then, $\succcurlyeq_{\hat{F}_1}$ is said to be a **partition-dependent (PD) unambiguity consistent extension** of $\succcurlyeq_{\hat{F}}$ to $\hat{F}_1$ if $\succcurlyeq_{\hat{F}_1}$ is $\mathcal{P}_1$-unambiguous where $\mathcal{P}_1 = \{E(P)\}_{P\in\mathcal{P}} \cup \{S_1^N\}$.

[24] Pires (2002) characterizes full-Bayesian updating in standard environments with fixed awareness. Although her axioms and our axioms are imposed on different objects, there are some similarities between the main characterizing axiom of Pires (2002) and our CEC. However, since CEC is not sufficient for reverse full-Bayesianism in general and we study growing awareness, her result does not imply our Theorem 3.

[25] Recall, under Unambiguity Consistency, $\{E_s\}_{s\in S} \cup \{S_1^N\}$ is the unambiguous partition according to $\succsim_{\hat{F}_1}$.

PD-Unambiguity Consistency is characterized as follows.

Proposition 1. *Suppose the initial and extended preference relations $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_1}$ are MEU preferences. Then $\succsim_{\hat{F}}$ satisfies $\mathcal{P}$-Independence and $\succsim_{\hat{F}_1}$ satisfies $\mathcal{P}_1$-Independence if and only if $\succsim_{\hat{F}}$ is $\mathcal{P}$-unambiguous and $\succsim_{\hat{F}_1}$ is a PD-unambiguity consistent extension of $\succsim_{\hat{F}}$.*

Proposition 1 shows that $\mathcal{E}$-Independence characterizes PD-Unambiguity Consistency. Note that $\mathcal{P}_1$-Independence is equivalent to the second part of NUI. Hence, Proposition 1 extends Theorem 1 to the case where both the initial and extended preferences are MEU.

5.3 Dynamic Consistency and Rectangularity

The objective of Sections 5.1-2 is to illustrate that Likelihood Consistency and Unambiguity Consistency can be extended to the setting where one MEU preference evolves to another MEU preference. In this section, we study rectangularity since it ensures dynamic consistency in MEU preferences, which is convenient for economic applications. Therefore, we characterize the cases in which in which the initial set of priors Π is $\mathcal{P}$-rectangular and the new set of priors Π_1 is $\mathcal{P}_1$-rectangular.

It turns out that rectangularity is characterized by the following straightforward weakening of Savage's Sure-Thing Principle.[26]

Definition 9 ($\mathcal{E}$-Separability)**.** For any partition $\mathcal{E}$ of S, a preference relation $\succsim$ on $\hat{F}$ satisfies **$\mathcal{E}$-Separability** if for any $E \in \mathcal{E}$ and $f, g, h, h' \in \hat{F}$, $fEh \sim gEh$ if and only if $fEh' \sim gEh'$.

We can now state our first result in which rectangularity and reverse full-Bayesianism are characterized simultaneously.

Theorem 4. *Suppose the initial preference relation $\succsim_{\hat{F}}$ is a MEU preference with (Π, U) and the extended preference relation $\succsim_{\hat{F}_1}$ is a MEU preference with (Π_1, U_1). The following are equivalent.*

(i) *$\succsim_{\hat{F}}$ satisfies $\mathcal{P}$-Separability, $\succsim_{\hat{F}_1}$ satisfies $\mathcal{P}_1$-Separability, and $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_1}$ jointly satisfy CEC and NCEI;*

(ii) *Π is $\mathcal{P}$-rectangular, Π_1 is $\mathcal{P}_1$-rectangular, $\succsim_{\hat{F}}$ is a reverse full-Bayesian extension of $\succsim_{\hat{F}_1}$, and there are α, β with $\alpha > 0$ such that $U_1(p) = \alpha\, U(p) + \beta$ for any $p \in \Delta(C)$.*

Similar to Corollary 5, $\mathcal{P}_1$-rectangularity does not require additive-separability under reverse full-Bayesianism as illustrated below.

Corollary 6. (Genuine expansion) Suppose that $|S_1^R| = |S|$. Let $\succsim_{\hat{F}}$ be a MEU preference with (Π, U) and $\succsim_{\hat{F}_1}$ be a MEU preference with (Π_1, U) as in Theorem 4. Then there is a

[26] Epstein and Schneider (2003) characterize rectangularity by dynamic consistency, which requires a consistency condition between the unconditional and conditional preference relations. We do not use the conditional preferences in this paper. Instead, we directly characterize rectangularity of Π (resp., Π_1) by an axiom on $\succsim_{\hat{F}}$ (resp., $\succsim_{\hat{F}_1}$).

set $[\underline{\delta},\overline{\delta}] \times \left(\prod_{P_1\in\mathcal{P}_1} \Pi^{P_1} \times \Pi^N\right) \subset [0,1] \times \prod_{P_1\in\mathcal{P}_1} \Delta(P_1) \times \Delta(S_1^N)$ such that for any $f \in \hat{F}_1$, $\min_{\pi\in\Pi_1}\sum_{s^1\in S_1} U\big(f(s^1)\big)\pi(s^1)$ is equal to

$$\min_{\delta\in[\underline{\delta},\overline{\delta}],\pi^{P_1}\in\Pi^{P_1}\,\forall P_1\in\mathcal{P}_1,\pi^N\in\Pi^N} \left\{\delta \sum_{P_1\in\mathcal{P}_1} \pi(P_1) \sum_{s^1\in P_1} U\big(f(s^1)\big)\pi^{P_1}(s^1) + (1-\delta)\sum_{s^1\in S_1^N} U\big(f(s^1)\big)\pi^N(s^1)\right\}.$$

In the next result, we characterize rectangularity and PD-Unambiguity Consistency simultaneously.

Proposition 2. *Suppose the initial and extended preference relations $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_1}$ are MEU preferences. The following are equivalent.*

(i) $\succsim_{\hat{F}}$ *satisfies $\mathcal{P}$-Separability and $\mathcal{P}$-Independence and $\succsim_{\hat{F}_1}$ satisfies $\mathcal{P}_1$-Separability and $\mathcal{P}_1$-Independence;*

(ii) Π *is $\mathcal{P}$-rectangular,* Π_1 *is $\mathcal{P}_1$-rectangular, $\succsim_{\hat{F}}$ is $\mathcal{P}$-unambiguous, and $\succsim_{\hat{F}_1}$ is a PD-unambiguous extension of $\succsim_{\hat{F}}$;*

(iii) $\succsim_{\hat{F}}$ *is $\mathcal{P}$-additive-separable and $\succsim_{\hat{F}_1}$ is $\mathcal{P}_1$-additive-separable.*

Unlike in Theorem 4, $\mathcal{P}_1$-rectangularity under PD-Unambiguity Consistency requires additive separability.

6 Parametric MEU for Refined Expansions

In this section, we go back to our initial setting in which the initial preference relation is a SEU preference and study a parametric version of the extended MEU preference that satisfies Unambiguity and Likelihood Consistency. The suggested parametric MEU model makes our theory tractable for broad economic applications and empirical studies.[27] For simplicity, we focus on refined expansions in which $S_1 = S_1^R = \bigcup_{s\in S} E_s$. Under Unambiguity Consistency and Likelihood Consistency, the extended MEU preference $\succsim_{\hat{F}_1}$ inherits the old beliefs, i.e., $\mu(s) = \pi(E_s)$ for each $s \in S$ and $\pi \in \Pi_1$.[28] We have argued that the DM perceives ambiguity because she may not know how to "split" her old belief $\mu(s)$ across the newly discovered states in E_s. In this section, we suggest the following procedural way to "split" $\mu(s)$.

Suppose that the DM considers a probability measure $\eta_s \in \Delta(E_s)$ on the new states in E_s. However, the DM is not confident that η_s "truthfully" describes the likelihoods of the newly discovered states. Therefore, the DM might distort η_s by a parameter $\alpha_s \in [0,1]$. For a given $\alpha_s \in [0,1]$ and $\eta_s \in \Delta(E_s)$, the DM forms beliefs over E_s defined as a convex mixture between η_s and the set

[27] Parametric versions of MEU model have been applied to asset pricing (Epstein and Wang, 1994), search theory (Nishimura and Ozaki, 2004), insurance models (Carlier et al., 2003), and mechanism design (Bose and Daripa, 2009).

[28] Recall that E_s is the event corresponding to the old state $s \in S$. In the KV approach, $E_s = \{s^1 \in S_1 : \exists c \in C \text{ s.t. } s^1 = (s,c)\}$.

of all possible probability measures in $\Delta(E_s)$. That is,

$$\Pi_{E_s}^{(\eta_s,\,\alpha_s)} = \alpha_s\{\eta_s\} + (1-\alpha_s)\Delta(E_s). \tag{11}$$

The parameter α_s might be interpreted as the DM's degree of confidence in the probability measure η_s. When $\alpha_s = 1$, the DM is confident that η_s accurately represents the likelihoods of the new states. When $\alpha_s = 0$, she is not confident at all and her beliefs are represented by the set of all priors $\Delta(E_s)$.[29]

Recall our patient example and Table 2. Suppose that the patient is told by her doctor that the probability that the novel treatment N is successful is 0.7. Therefore, she considers $\eta_{s_1} = (\eta(s_1^1), \eta(s_5^1)) = (0.7, 0.3)$ as a probability measure over $E_{s_1} = \{s_1^1, s_5^1\}$. When her degree of confidence in η_{s_1} is α_1, her beliefs over E_{s_1} are

$$\Pi_{E_{s_1}}^{(\eta_1,\,\alpha_1)} = \alpha_1\,\eta_1 + (1-\alpha_1)\,\Delta(E_{s_1}) = \left\{\left(\pi(s_1^1), \pi(s_5^1)\right) : \pi(s_1^1) \in [0.7\,\alpha_1, 1-0.3\,\alpha_1]\right\}. \tag{12}$$

We allow η_s to depend on E_s as well as the DM's degree of confidence α_s in η_s to vary across $s \in S$. In other words, different degrees of confidence α_s might reflect her perception that each original state $s \in S$ is affected differently by the discovery of $\bar{f}$. For example, suppose that the patient also considers $\eta_4 = (0.7, 0.3)$ on E_{s_4}. However, the patient might be more cautious about success of the novel treatment $\bar{f}$ when she considers event E_{s_4}, in which the standard treatments f_1 and f_2 fail, as compared to event E_{s_1} in which f_1 and f_2 are successful. Therefore, her degree of confidence α_1 might be larger than α_4.

Our parametric MEU representation of the extended preference $\succcurlyeq_{\hat{F}_1}$ takes the following form. For a given parameter vector $\{(\eta_s, \alpha_s)\}_{s\in S} \in \prod_{s\in S}(\Delta(E_s)\times[0,1])$, each act $f \in \hat{F}_1$ is evaluated via

$$V(f) = \sum_{s\in S}\mu(s)\Big(\min_{\pi\in\Pi_{E_s}^{(\eta_s,\alpha_s)}}\sum_{s^1\in E_s}U\big(f(s^1)\big)\pi(s^1)\Big), \tag{13}$$

where the set of priors $\Pi_{E_s}^{(\eta_s,\,\alpha_s)}$ for each $s \in S$ is defined in Equation (11). This parametric MEU preference satisfies both Unambiguity Consistency and Likelihood Consistency. Moreover, the representation admits an additive decomposition across the unambiguous events $\{E_s\}_{s\in S}$. Theorem 5 in Appendix B axiomatically characterizes this parametric MEU representation by BAC and a stronger version of our key axiom, NUI.

6.1 Parametric MEU for Genuine Expansions

Let us conclude this section by briefly discussing a parametric MEU representation in the context of genuine expansions. Similar to Theorem 5 in Appendix B, we can obtain the following parametric

[29] A similar procedure has been studied in the literature. For example, Kopylov (2016) axiomatizes a MEU representation with a set of priors $\Pi_1 = \alpha\,\eta + (1-\alpha)\Delta(S_1)$, where $\eta \in \Delta(S_1)$ for a given state space S_1. In our setting, we have α_s and η_s for each event $E_s \subset S_1$ and allow for $\alpha_s \neq \alpha_{s'}$.

representation: there exist $\delta, \alpha \in [0,1]$ and $\eta \in \Delta\big(S_1^N\big)$ such that every act $f \in \hat{F}_1$ is evaluated via the functional:

$$V(f) = \delta\Big(\sum_{s\in S}\mu(s)U(f(E_s))\Big) + (1-\delta)\Big(\min_{\pi\in\Pi^{(\eta,\,\alpha)}}\sum_{s^1\in S_1^N}U\big(f(s^1)\big)\pi(s^1)\Big),$$

where $\Pi^{(\eta,\,\alpha)} = \alpha\,\{\eta\} + (1-\alpha)\,\Delta\big(S_1^N\big)$.

7 Related Literature

There are two main approaches to modelling unawareness and growing awareness in the economic literature: the preference-based approach and the epistemic approach.[30] The preference-based approach, which we follow, assumes an exogenously given unawareness structure and studies how unawareness and growing awareness may affect choice under uncertainty. The first rigorous study in this context is Karni and Vierø (2013). KV develop the theory of reverse Bayesianism that characterizes a likelihood-consistent evolution of SEU preferences under growing awareness due to discoveries of new acts, new consequences, or links between them.[31]Karni and Vierø (2015) extend their reverse Bayesianism to probabilistically sophisticated preferences of Machina and Schmeidler (1992, 1995). The above two papers and in fact most of the literature on choice under growing awareness focus on preferences with probabilistic beliefs. Our paper goes beyond this paradigm since ambiguity averse behavior is inconsistent with probabilistically sophisticated preferences. In fact, we illustrate that growing awareness of states is a novel source of ambiguity and ambiguity averse behavior. We also introduce and characterize two consistency notions, Unambiguity Consistency and Likelihood Consistency, and reverse full-Bayesianism, which is an extension of the reverse Bayesianism of KV to MEU preferences.[32]

Grant et al. (2018) study ambiguity under growing awareness from a different prospective by developing a model of learning under unawareness. In their model, a DM has incomplete information about the structure of the state space. She learns about the unknown states through sequential experimentation. At the initial stage, the DM is completely ignorant and her beliefs are represented by the set of all priors. The DM's beliefs are successively updated while discovering new acts and new consequences. Grant et al. (2018) and our paper complement each other. We axiomatically characterize two consistency notions for belief updating under growing awareness and ambiguity while Grant et al. (2018) study the underlying stochastic process of learning and belief updating using the imprecise Dirichlet process.

[30]The goal of the epistemic approach is to develop formal models of unawareness and to study logical properties of unawareness. Schipper (2014a,b) provide comprehensive surveys of the literature on the epistemic approach to unawareness.

[31]In a recent study, Chakravarty et al. (2019) apply the theory of reverse Bayesianism of KV to explore the implications of unawareness and growing awareness for the economic analysis of tort law.

[32]Hayashi (2012) studies the evolution of subjective probabilities from the point of view of dynamic behavior. In his setup, the state space expansion follows a product structure. By imposing a form of dynamic consistency between choices made before and after a state space expansion, he characterizes a consistent evolution of beliefs in the sense that the marginal distribution of the new belief induced over the old state space coincides with the old belief.

Our theory and the models mentioned above preclude awareness of unawareness. Karni and Vierø (2017) provide an extension of reverse Bayesianism to situations in which a DM is aware that she is unaware of some consequences. That is, the DM anticipates discovery of unknown consequences, and assigns utilities to unspecified consequences even if such consequences may not even exist. Under the assumption of SEU preferences, they characterize the principle of reverse Bayesianism in this context.[33]

There is another strand of literature that focuses on situations where a DM has a limited understanding of a given state space or a feasible set of acts rather than being unaware of them. The primary goal is to provide an alternative representation of preferences but not to characterize a consistent evolution of preferences under, e.g., improving understating of the choice environment.

Ahn and Ergin (2010) derive a partition-dependent SEU representation in which the DM's beliefs depend on descriptions of states, which are represented by partitions of a given state space. Although the DM may receive better descriptions through refining the original partition, the refinement process is tacit and the new preferences are not linked to the old preferences under coarser partitions. Moreover, the SEU form of preferences does not change as the DM's understanding improves.

Lehrer and Teper (2014) study rules that extend restricted complete SEU preferences (defined over a restricted set of acts) to unrestricted but incomplete preferences (defined over the entire domain of acts). Under the so-called prudent rule, the extended preferences are incomplete à la Bewley (2002). They also discuss how to complete the Bewley representation and, under a modified prudent rule, the completion takes a restricted MEU form. Similar to us, Lehrer and Teper (2014) allow preferences to change as the set of acts expands. Besides that, there are several substantial differences from our approach. First, in their setup the evolution of beliefs is not addressed while our theory characterizes consistent evolution of beliefs and tastes. Second, in their setup the state space is intact. That is, although the set of acts expands, it does not affect the description of the original states. Finally, the set of priors takes a particular form of complete ignorance and the existence of such a set is not triggered by new discoveries per se. In particular, the set of priors is induced by the initial preference relation; it is the set of all probability measures that rationalizes the DM's (original) preference over the restricted set of acts.

Another model that allows for ambiguity is Grant and Quiggin (2015). In their model, a DM might be surprised by an unforeseen (monetary) consequence. A state space is augmented by "surprise" states in which either an unfavorable consequence ranked below the worst possible consequence or a favorable consequence ranked above the best possible consequence might arise. They derive a representation of preferences that captures the DM's aversion towards unfavorable surprises and proneness towards favorable surprises. When surprise states are viewed as impossible, preferences admit an expected uncertain utility representation of Gul and Pesendorfer (2014).

[33] Alon (2015) also derives a choice model in which the DM is aware of her unawareness. The DM's unawareness is represented by an imaginary, "unforeseen event," extending the exogenous state space. While evaluating an act, the imaginary event is associated with the worst possible consequence leading to the so-called worst-case expected utility representation.

A Appendix: Proofs

A.1 A Useful Lemma for Theorems 1-2

In Theorems 1-2, we need to show that $\succcurlyeq_{\hat{F}_1}$ admits a MEU representation. Since $\succcurlyeq_{\hat{F}_1}$ satisfies (A.1)-(A.4), it is suffices to show that $\succcurlyeq_{\hat{F}_1}$ satisfies the following two key axioms of Gilboa and Schmeidler (1989):

(A.12) (**Certainty Independence**) For all $f, g \in \hat{F}_1$, $c \in C_1$, and $\alpha \in (0,1]$, $f \succcurlyeq_{\hat{F}_1} g$ if and only if $\alpha f + (1-\alpha)c \succcurlyeq_{\hat{F}_1} \alpha g + (1-\alpha)c$.

(A.13) (**Uncertainty Aversion**) For all $f, g \in \hat{F}_1$, $f \succcurlyeq_{\hat{F}_1} g$ if and only if $\alpha f + (1-\alpha)g \succcurlyeq_{\hat{F}_1} g$ for all $\alpha \in (0,1]$.

Lemma 2 shows that WNUI implies Certainty Independence and Uncertainty Aversion.

Lemma 2. *If $\succcurlyeq_{\hat{F}_1}$ satisfies (A.1)-(A.4) and WNUI, then it satisfies (A.11)-(A.12).*

Proof of Lemma 2. Suppose $\succcurlyeq_{\hat{F}_1}$ satisfies (A.1)-(A.4) and WNUI. First, we prove that Certainty Independence is satisfied. Take any $f, g \in \hat{F}_1$, $c \in C_1$, and $\alpha \in (0,1]$ with $f \succcurlyeq_{\hat{F}_1} g$. Moreover, take a lottery $q \in \Delta(C_1)$ such that $g \sim_{\hat{F}_1} q$. Weak NUI implies that for any $\alpha \in [0,1]$,

$$\text{if} \;\; f \succcurlyeq_{\hat{F}_1} q, \;\; \text{then} \;\; \alpha f + (1-\alpha)c \succcurlyeq_{\hat{F}_1} \alpha q + (1-\alpha)c$$

and

$$g \sim_{\hat{F}_1} q \quad \text{if and only if} \quad \alpha g + (1-\alpha)c \sim_{\hat{F}_1} \alpha q + (1-\alpha)c.$$

Therefore, by Transitivity, $f \succcurlyeq_{\hat{F}_1} g$ implies $\alpha f + (1-\alpha)c \succcurlyeq_{\hat{F}_1} \alpha g + (1-\alpha)c$. The opposite direction of Certainty Independence is obvious.

Second, we will prove that Uncertainty Aversion is satisfied. Take any $f, g \in \hat{F}_1$ and $\alpha \in (0,1]$ with $f \succcurlyeq_{\hat{F}_1} g$. Moreover, take a lottery $q \in \Delta(C_1)$ such that $g \sim_{\hat{F}_1} q$. Weak NUI implies that for any $\alpha \in [0,1]$,

$$\text{if} \;\; f \succcurlyeq_{\hat{F}_1} q, \;\; \text{then} \;\; \alpha f + (1-\alpha)g \succcurlyeq_{\hat{F}_1} \alpha q + (1-\alpha)g.$$

By Certainty Independence, $\alpha g + (1-\alpha)q \sim_{\hat{F}_1} \alpha q + (1-\alpha)q = q$. Therefore, by Transitivity, if $f \succcurlyeq_{\hat{F}_1} q$, then $\alpha f + (1-\alpha)g \succcurlyeq_{\hat{F}_1} g$. The opposite direction of Uncertainty Aversion is also immediate. This completes the proof.

Since $\succcurlyeq_{\hat{F}}$ satisfies axioms (A.1)-(A.5) and $\succcurlyeq_{\hat{F}_1}$ satisfies (A.1)-(A.4) and WNUI in Theorems 1-2, from now we assume that $\succcurlyeq_{\hat{F}}$ has a SEU representation and $\succcurlyeq_{\hat{F}_1}$ has a MEU representation. That is, there exist a non-constant and affine function $U : \Delta(C) \to \mathbb{R}$, and a probability measure μ on S, such that $\succcurlyeq_{\hat{F}}$ admits the representation (1) with (μ, U) and there exist a non-constant and affine function $U_1 : \Delta(C_1) \to \mathbb{R}$, a convex and compact set of probability measures $\Pi_1 \subseteq \Delta(S_1)$, such that $\succcurlyeq_{\hat{F}_1}$ admits the representation (2) with (Π_1, U_1). The uniqueness of U, U_1, μ, and Π_1 are straightforward. Since the necessity parts of Theorems 1-2 are straightforward, we only prove their sufficiency parts.

A.2 Proof of Theorem 1

Suppose $\succcurlyeq_{\hat{F}}$ satisfies axioms (A.1)-(A.5), $\succcurlyeq_{\hat{F}_1}$ satisfies axioms (A.1)-(A.4) and NUI. By the discussion in Section A.1, suppose $\succcurlyeq_{\hat{F}}$ has a SEU representation with (μ, U) and $\succcurlyeq_{\hat{F}_1}$ has a MEU representation with (Π_1, U_1). Without loss of generality, let $U_1(b) = 1$ and $U_1(w) = 0$ where b and w are the best and worst consequences in C_1, respectively.

Finally, we shall prove that $\succcurlyeq_{\hat{F}_1}$ is an unambiguity consistent extension of $\succcurlyeq_{\hat{F}}$ to $\hat{F}_1$. Let us fix $s \in S$. NUI implies that for all $f, g \in \hat{F}_1$ with $g = p_{E_s} w$ for some $p \in \Delta(C_1)$ and for all $q \in \Delta(C_1)$ and $\alpha \in [0,1]$, if $f \sim_{\hat{F}_1} q$, then $\alpha f + (1-\alpha) g \sim_{\hat{F}_1} \alpha q + (1-\alpha) g$. In terms of the MEU representation (2) for $\succcurlyeq_{\hat{F}_1}$, if

$$\min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U_1\big(f(s^1)\big)\pi(s^1) = U_1(q), \text{ then} \tag{14}$$

$$\min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} \big(\alpha\, U_1\big(f(s^1)\big) + (1-\alpha) U_1\big(g(s^1)\big)\big)\, \pi(s^1) = \alpha\, U_1(q) + (1-\alpha) \min_{\pi \in \Pi_1} \sum_{s^1 \in S_1} U_1\big(g(s^1)\big)\pi(s^1). \tag{15}$$

Since $g = p_{E_s} w$, Equation (15) is equivalent to

$$\min_{\pi \in \Pi_1} \left\{ \alpha \sum_{s^1 \in S_1} U_1\big(f(s^1)\big)\pi(s^1) + (1-\alpha)\, \pi(E_s)\, U_1(p) \right\} = \alpha\, U_1(q) + (1-\alpha) \min_{\pi \in \Pi_1} \left\{ \pi(E_s) U_1(p) \right\}.$$

Therefore, by combining (14) and (15), we have that for any $f \in \hat{F}_1$, $p \in \Delta(C)$, and $\alpha \in [0,1]$,

$$\begin{aligned} &\min_{\pi \in \Pi_1} \left\{ \alpha \sum_{s^1 \in S_1} U_1\big(f(s^1)\big)\pi(s^1) + (1-\alpha)\pi(E_s)\, U_1(p) \right\} \\ &= \alpha \min_{\pi \in \Pi_1} \left\{ \sum_{s^1 \in S_1} U_1\big(f(s^1)\big)\pi(s^1) \right\} + (1-\alpha) \min_{\pi \in \Pi_1} \left\{ \pi(E_s) \right\} U_1(p). \end{aligned} \tag{16}$$

Let us now assume that $\alpha = \frac{1}{2}$ and $f = q_1 E_s p$ for some $q_1 \in \Delta(C_1)$. Then (16) is equivalent to

$$\min_{\pi \in \Pi_1} \left\{ \pi(E_s) \right\} U_1(q_1) + U_1(p) = \min_{\pi \in \Pi_1} \left\{ \pi(E_s) U_1(q_1) + (1 - \pi(E_s) U_1(p)) \right\} + \min_{\pi \in \Pi_1} \left\{ \pi(E_s) \right\} U_1(p).$$

Suppose that $U_1(p) > U_1(q_1)$. Then the above equality is equivalent to

$$\left(\max_{\pi \in \Pi_1} \left\{ \pi(E_s) \right\} - \min_{\pi \in \Pi_1} \left\{ \pi(E_s) \right\} \right) U_1(q_1) = \left(\max_{\pi \in \Pi_1} \left\{ \pi(E_s) \right\} - \min_{\pi \in \Pi_1} \left\{ \pi(E_s) \right\} \right) U_1(p). \tag{17}$$

Since Equation (17) is satisfied for any $q_1, p \in \Delta(C)$ with $U_1(p) > U_1(q_1)$, we have that $\min_{\pi \in \Pi_1} \left\{ \pi(E_s) \right\} = \max_{\pi \in \Pi_1} \left\{ \pi(E_s) \right\}$, i.e., E_s is an unambiguous event.

A.3 Proof of Theorem 2

Suppose $\succcurlyeq_{\hat{F}}$ satisfies axioms (A.1)-(A.5), $\succcurlyeq_{\hat{F}_1}$ satisfies axioms (A.1)-(A.4), and WNUI, and they jointly satisfy BAC. By the discussion in Section A.1, suppose $\succcurlyeq_{\hat{F}}$ has a SEU representation with (μ, U) and $\succcurlyeq_{\hat{F}_1}$ has a SEU representation with (Π_1, U_1). Without loss of generality, let $U(b) = U_1(b) = 1$ and $U(w) = U_1(w) = 0$ where b and w are the best and worst consequences in C, respectively. We prove Theorem 2 in the following two steps.

Step 1: $U = U_1$ on $\Delta(C)$.

For any $p, p' \in \Delta(C)$, by BAC, we have $(p_{-s}\, p) \succcurlyeq_{\hat{F}} (p'_{-s}\, p')$ if and only if $\big(w_{-E(S)}(p_{-E_s}\, p)\big) \succcurlyeq_{\hat{F}_1} \big(w_{-E(S)}(p'_{-E_s}\, p')\big)$; equivalently,

$$U\big(p\big) \geq U\big(p'\big) \text{ if and only if } \min_{\pi\in\Pi_1} \{\pi(E(S))\}\, U_1(p) \geq \min_{\pi\in\Pi_1} \{\pi(E(S))\}\, U_1(p') \text{ if and only if } U_1\big(p\big) \geq U_1\big(q\big).$$

Therefore, $U = U_1$ on $\Delta(C)$, i.e, risk attitudes of $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_1}$ are the same.

Step 2: $\succcurlyeq_{\hat{F}_1}$ is a likelihood consistent extension of $\succcurlyeq_{\hat{F}}$.

For any $s \in S$, $p, q, p' \in \Delta(C)$, by BAC, we have

$$(p_{-s}\, q) \succcurlyeq_{\hat{F}} (p'_{-s}\, p') = p' \;\text{ if and only if }\; \big(p'_{-E(S)}(p_{-E_s}\, q)\big) \succcurlyeq_{\hat{F}_1} \big(p'_{-E(S)}(p'_{-E_s}\, p')\big) = p';$$

equivalently,

$$(1-\mu(s))U(p) + \mu(s)U(q) = U\big(p'\big) \text{ if and only if}$$

$$\min_{\pi\in\Pi_1} \big\{\pi(E(S)\backslash E_s)U(p) + \pi\big(E_s\big)U(q) + (1 - E(S))U\big(p'\big)\big\} = U(p').$$

Therefore, we have for any $s \in S$ and $p, q \in \Delta(C)$, $(1-\mu(s))U(p) + \mu(s)U(q)$ is equal to

$$\min_{\pi\in\Pi_1} \big\{\pi(E(S)\backslash E_s)U(p) + \pi(E_s)U(q) + \big(1 - \pi(E(S))\big)\big((1-\mu(s))U(p) + \mu(s)U(q)\big)\big\}.$$

Therefore, for any $s \in S$ and $p, q \in \Delta(C)$,

$$\min_{\pi\in\Pi_1} \big\{\big(\pi(E_s) - \pi(E(S))\mu(s)\big)\big(U(q) - U(p)\big)\big\} = 0.$$

The above equation implies $\min_{\pi\in\Pi_1} \{\pi(E_s) - \pi(E(S))\mu(s)\} = 0$ when $U(p) < U(q)$ and $\max_{\pi\in\Pi_1} \{\pi(E_s) - \pi(E(S))\mu(s)\} = 0$ when $U(p) > U(q)$. Therefore, $\pi(E_s) = \pi(E(S))\mu(s)$ for any $s \in S$ and $\pi \in \Pi_1$.

A.4 Proof of Theorem 3

Suffiency. We prove the sufficiency part of Theorem 3 by six steps.

Step 1. There are α, β with $\alpha > 0$ such that $U_1(p) = \alpha\, U(p) + \beta$ for any $p \in \Delta(C)$.

Take any $f \in \hat{F}$ and $p \in \Delta(C)$ with $f \sim_{\hat{F}} p$; equivalently,

$$\min_{\pi\in\Pi}\sum_{s\in S} U\big(f(s)\big)\pi(s) = U(p). \tag{18}$$

By CEC, we have $(f(s)\, E_s)_{s\in S} \cup p \sim_{\hat{F}_1} p$; equivalently,

$$\min_{\pi_1\in\Pi_1}\{\sum_{s\in S} U_1\big(f(s)\big)\pi_1(E_s) + (1-\sum_{s\in S}\pi_1(E_s))U_1(p)\} = U_1(p). \tag{19}$$

Let $f(s) = q$ for some $q \in \Delta(C)$. Then $U(p) = U(q)$ if and only if $U_1(p) = U_1(q)$. Hence, there are α, β with $\alpha > 0$ such that $U_1(p) = \alpha\, U(p) + \beta$ for any $p \in \Delta(C)$.

Step 2. For any extreme point $\bar{\pi}$ of Π, there is $\pi_1 \in \Pi_1$ such that for any $s \in S$,

$$\bar{\pi}(s) = \frac{\pi_1(E_s)}{\sum_{s'\in S}\pi_1(E_{s'})}.$$

Take any extreme point $\bar{\pi}$ in Π. There is $f \in \hat{F}$ such that $\bar{\pi}$ be the minimizer of (18) and $U(f(s)) \neq U(f(s'))$ for any $s, s' \in S$. Given f, let $\bar{\pi}_1$ be the minimizer of (19). Combining (18) and (19), we have

$$\sum_{s\in S} U\big(f(s)\big)\big(\bar{\pi}_1(E_s) + (1-\sum_{s\in S}\bar{\pi}_1(E_s))\bar{\pi}(s)\big) = \sum_{s\in S} U\big(f(s)\big)\bar{\pi}(s). \tag{20}$$

We shall show that $\bar{\pi}_1(E_s) + (1-\sum_{s\in S}\bar{\pi}_1(E_s))\bar{\pi}(s) = \bar{\pi}(s)$ for each $s \in S$. Take any $s \in S$ and take another act f' such that $f'(s') = f(s')$ for any $s' \in S \setminus \{s\}$ and $|U(f'(s)) - U(f(s))| = \epsilon > 0$. When ϵ is small enough, $\bar{\pi}$ and $\bar{\pi}_1$ are still the minimizers of (18) and (19) for f'. By (20), we will have

$$\sum_{s\in S} U\big(f'(s)\big)\big(\bar{\pi}_1(E_s) + (1-\sum_{s\in S}\bar{\pi}_1(E_s))\bar{\pi}(s)\big) = \sum_{s\in S} U\big(f'(s)\big)\bar{\pi}(s). \tag{21}$$

Subtracting (21) from (20), we will have $(U\big(f(s)\big) - U(f'(s)))\big(\bar{\pi}_1(E_s) + (1-\sum_{s\in S}\bar{\pi}_1(E_s))\bar{\pi}(s) - \bar{\pi}(s)\big) = 0$. Hence, $\bar{\pi}_1(E_s) + (1-\sum_{s\in S}\pi_1(E_s))\bar{\pi}(s) = \bar{\pi}(s)$.

Notice that we just proved that $\bar{\pi}(s) = \frac{\bar{\pi}_1(E_s)}{\sum_{s\in S}\pi_1(E_s)}$ for any $s \in S$. Equivalently, for any extreme point $\pi \in \Pi$, there is $\pi_1 \in \Pi_1$ such that

$$\frac{\bar{\pi}(s)}{\bar{\pi}(s')} \quad = \quad \frac{\bar{\pi}_1(E_s)}{\bar{\pi}_1(E_{s'})} \text{ for all } s, s' \in S. \tag{22}$$

Step 3. For any $\pi \in \Pi$, there is $\pi_1 \in \Pi_1$ such that for any $s \in S$,

$$\pi(s) = \frac{\pi_1(E_s)}{\sum_{s'\in S}\pi_1(E_{s'})}.$$

Take any $\pi \in \Pi$. Since Π is a convex set with finite extreme points, there are sets of extreme points $\{\pi^i\}_{i=1}^m$ in Π and strictly positive weights $\{\alpha^i\}_{i=1}^m$ such that $\pi = \sum_{i=1}^m \alpha^i \pi^i$ and $\sum_{i=1}^m \alpha^i = 1$. By Step 2, there is a set $\{\pi_1^i\}_{i=1}^m$ in Π_1 such that $\pi^i(s) = \frac{\pi_1^i(E_s)}{\sum_{s' \in S} \pi_1^i(E_{s'})}$ for each $s \in S$ and $i \leq m$. We now show that there is a set of strictly positive weights $\{\beta^i\}_{i=1}^m$ such that $\pi(s) = \frac{\pi_1(E_s)}{\sum_{s' \in S} \pi_1(E_{s'})}$ and $\sum_{i=1}^m \beta^i = 1$ where $\pi_1 = \sum_{i=1}^m \beta^i \pi_1^i$. Let $\beta^i \equiv \lambda \frac{\alpha^i}{\pi_1^i(E(S))}$ where $\lambda \equiv \frac{1}{\sum_{i=1}^m \frac{\alpha_i}{\pi_1^i(E(S))}}$. Then

$$\frac{\pi_1(E_s)}{\sum_{s' \in S} \pi_1(E_{s'})} = \frac{\sum_{i=1}^m \beta^i \pi_1^i(E_s)}{\sum_{s' \in S} \sum_{i=1}^m \beta^i \pi_1^i(E_{s'})} = \frac{\sum_{i=1}^m \beta^i \pi_1^i(E_s)}{\sum_{i=1}^m \beta^i \pi_1^i(E(S))}$$

$$= \frac{\sum_{i=1}^m \lambda \frac{\alpha^i}{\pi_1^i(E(S))} \pi_1^i(E_s)}{\sum_{i=1}^m \lambda \frac{\alpha^i}{\pi_1^i(E(S))} \pi_1^i(E(S))} = \sum_{i=1}^m \alpha^i \frac{\pi_1^i(E_s)}{\sum_{s' \in S} \pi_1^i(E_{s'})} = \sum_{i=1}^m \alpha^i \, \pi^i(s) = \pi(s).$$

To state Step 4, some notations are necessary. Let $S^* \equiv S \cup \{s^*\}$ and

$$\Pi^* \equiv \{\pi^* \in \Delta(S^*) | \exists \pi_1 \in \Pi_1 \text{ such that } \pi^*(s) = \pi_1(E_s) \text{ for all } s \in S \text{ and } \pi^*(s^*) = \pi_1(S_1^N)\}.$$

Step 3 proves that $\Pi \subseteq \Pi^*|_S$. Hence, the remainder of this proof shows that $\Pi^*|_S \subseteq \Pi$.

Step 4. If $\Pi^*|_S \setminus \Pi \neq \emptyset$, then there is an extreme point $\bar{\pi}^*$ of Π^* such that $\bar{\pi}^{**} = (\frac{\pi^*(s)}{\sum_{s' \in S} \pi^*(s')})_{s \in S}$ is an extreme point of $\Pi^*|_S$ and $\pi^{**} \notin \Pi$.

Take any extreme point π of $\Pi^*|_S$ such that $\pi \notin \Pi$. Suppose that $\pi = \pi^*|_S$ for some $\pi^* \in \Pi^*$. If π^* is an extreme point of Π^*, then we obtain the desired result. If π^* is not an extreme point of Π^*, then there are sets of extreme points $\{\pi^i\}_{i=1}^m$ in Π^* and strictly positive weights $\{\beta^i\}_{i=1}^m$ such that $\pi^* = \sum_{i=1}^m \beta^i \pi^i$ and $\sum_{i=1}^m \beta^i = 1$. By essentially identical argument in the proof of Step 3, we have $\pi = \pi^*|_S = \sum_{i=1}^m \alpha^i \pi^i|_S$ where $\alpha^i = \frac{\beta^i \, \pi^i(S)}{\sum_{j=1}^m \beta^j \, \pi^j(S)}$.[34] However, $\pi = \sum_{i=1}^m \alpha^i \pi^i|_S$ contradicts the fact that π is an extreme point of $\Pi^*|_S$.

Step 5. Take any extreme point $\bar{\pi}^*$ of Π^*. There are $f \in \hat{F}$ and $q \in \Delta(C)$ such that $\bar{\pi}^*$ is the solution to $\min_{\pi^* \in \Pi^*}\{\sum_{s \in S} U\big(f(s)\big)\pi^*(s) + U(q)\pi^*(s^*)\}$ where either $f(s) \succ_{\hat{F}} q$ for each $s \in S$, $q \succ_{\hat{F}} f(s)$ for each $s \in S$, or $f \sim_{\hat{F}} q$.

Indeed, there are $f' \in \hat{F}$ and $q \in \Delta(C)$ such that $\bar{\pi}^* \in \arg\min_{\pi^* \in \Pi^*}\{\sum_{s \in S} U\big(f'(s)\big)\pi^*(s) + U(q)\pi^*(s^*)\}$ and $U(f'(s)) \neq U(f'(s'))$ and $U(f'(s)) \neq U(q)$ for any $s, s' \in S$. If either $f'(s) \succ_{\hat{F}} q$ for each $s \in S$ or $q \succ_{\hat{F}} f'(s)$ for each $s \in S$, then we obtain the desired result. Suppose now $\max_{s \in S} U(f'(s)) > U(q) > \min_{s \in S} U(f'(s))$. Let f_λ be an act in $\hat{F}$ such that $f_\lambda(s) = \lambda f'(s) + (1 - \lambda)\, q$ when $U(f'(s)) > U(q)$ and $f_\lambda(s) = f'(s)$ when $U(f'(s)) < U(q)$. Since $f' = f_1 \succ_{\hat{F}} q \succ_{\hat{F}} f_0$, there is λ^* such that $f_{\lambda^*} \sim_{\hat{F}} q$. Since f_{λ^*} and f' are comonotonic, $\bar{\pi}^*$ is also the minimizer of $\min_{\pi^* \in \Pi^*}\{\sum_{s \in S} U\big(f_{\lambda^*}(s)\big)\pi^*(s) + U(q)\pi^*(s^*)\}$.

Step 6. $\Pi^*|_S = \Pi$.

[34] In the proof of Step 3, we construct $\{\beta^i\}_{i=1}^m$ given $\{\alpha^i\}_{i=1}^m$. It is easy to notice that $\{\alpha^i\}_{i=1}^m$ can be found from $\{\beta^i\}_{i=1}^m$ instead.

By way of contradiction, suppose $\Pi^*|_S \setminus \Pi \neq \emptyset$. By Step 4, there is an extreme point $\bar{\pi}^*$ of Π^* such that $\bar{\pi}^{**} = (\frac{\pi^*(s)}{\sum_{s' \in S} \pi^*(s')})_{s \in S}$ is an extreme point of $\Pi^*|_S$ and $\pi^{**} \notin \Pi$.

Step 6.1. By Step 5, there are $f \in \hat{F}$ and $q \in \Delta(C)$ such that $\bar{\pi}^* \in \arg\min_{\pi^* \in \Pi^*}\{\sum_{s \in S} U\big(f(s)\big)\pi^*(s) + U(q)\pi^*(s^*)\}$ and either $f(s) \succ_{\hat{F}} q$ for each $s \in S$, $q \succ_{\hat{F}} f(s)$ for each $s \in S$, or $f \sim_{\hat{F}} q$. If $f \sim_{\hat{F}} q$, then by repeating Step 2, we can prove that

$$\bar{\pi}(s) = \frac{\bar{\pi}^*(s)}{\sum_{s' \in S} \bar{\pi}^*(s')},$$

where $\bar{\pi}$ is the minimizer of (18) for f. We now assume that either $f(s) \succ_{\hat{F}} q$ for each $s \in S$ or $q \succ_{\hat{F}} f(s)$ for each $s \in S$.

Take $p \in \Delta(C)$ such that $f \sim_{\hat{F}} p$. By NCEI, we have $(f(s)\, E_s)_{s \in S} \cup q \succsim_{\hat{F}_1} p\, E(S) q$; equivalently,

$$\sum_{s \in S} U\big(f(s)\big)\bar{\pi}^*(s) + (1 - \sum_{s \in S} \bar{\pi}^*(s))U(q) \geq (\sum_{s \in S} \tilde{\pi}^*(s))U(p) + (1 - \sum_{s \in S} \tilde{\pi}^*(s))U(q),$$

where $\tilde{\pi}^*$ minimizes $\min_{\pi^* \in \Pi^*}\{(\sum_{s \in S} \pi^*(s))U(p) + (1 - \sum_{s \in S} \pi^*(s))U(q)\}$.

When $q \succ_{\hat{F}} f(s)$ for each $s \in S$, we have $\sum_{s \in S} \tilde{\pi}^*(s) = \sum_{s \in S} \bar{\pi}^*(s) = \min_{\pi^* \in \Pi^*} \sum_{s \in S} \pi^*(s)$. Hence, we will have

$$\sum_{s \in S} U\big(f(s)\big) \frac{\bar{\pi}^*(s)}{\sum_{s \in S} \bar{\pi}^*(s)} \geq U(p).$$

When $f(s) \succ_{\hat{F}} q$ for each $s \in S$, we have $\sum_{s \in S} \tilde{\pi}^*(s) = \sum_{s \in S} \bar{\pi}^*(s) = \max_{\pi^* \in \Pi^*} \sum_{s \in S} \pi^*(s)$. Hence, we will have

$$\sum_{s \in S} U\big(f(s)\big) \frac{\bar{\pi}^*(s)}{\sum_{s \in S} \bar{\pi}^*(s)} \geq U(p).$$

In each case, since $f \sim_{\hat{F}} p$ is equivalent to $\min_{\pi \in \Pi} \sum_{s \in S} U\big(f(s)\big)\pi(s) = U(p)$, we will have

$$\sum_{s \in S} U\big(f(s)\big) \frac{\bar{\pi}^*(s)}{\sum_{s \in S} \bar{\pi}^*(s)} \geq \min_{\pi \in \Pi} \sum_{s \in S} U\big(f(s)\big)\pi(s). \tag{23}$$

Step 6.2. Step 3 indirectly proves that $\Pi^*|_S$ is convex. Let $\Pi^{**} = \text{co}(\Pi^*|_S \cup \Pi)$. Since π^{**} is an extreme point of $\Pi^*|_S$ and $\pi^{**} \notin \Pi$, π^{**} is an extreme point of Π^{**}. Then there is an act $f' \in \hat{F}$ such that

$$\sum_{s \in S} U\big(f'(s)\big)\pi^{**}(s) < \sum_{s \in S} U\big(f'(s)\big)\pi(s) \text{ for any } \pi \in \Pi^{**} \setminus \{\pi^{**}\}. \tag{24}$$

Now take $q' \in \Delta(C)$ such that $f'(s) \succ_{\hat{F}} q'$ for each $s \in S$ if $f(s) \succ_{\hat{F}} q$ for each $s \in S$ and $q' \succ_{\hat{F}} f'(s)$ for each $s \in S$ if $q \succ_{\hat{F}} f(s)$ for each $s \in S$. Now notice that $\bar{\pi}^* \in \arg\min_{\pi^* \in \Pi^*}\{\sum_{s \in S} U\big(f'(s)\big)\pi^*(s) + U(q')\pi^*(s^*)\}$. This is because the minimizer should assign $\bar{\pi}^*(s^*) = \min_{\pi^* \in \Pi^*} \sum_{s \in S} \pi^*(s^*)$ to state s^* when $q \succ_{\hat{F}} f(s)$ for each $s \in S$ (similarly, $\bar{\pi}^*(s^*) = \min_{\pi^* \in \Pi^*} \sum_{s \in S} \pi^*(s^*)$ when $q \prec_{\hat{F}} f(s)$ for each $s \in S$) and allocate probability $\sum_{s\in S} \bar{\pi}^*(s)$ between states in S to minimize $\sum_{s\in S} U\big(f'(s)\big)\pi^*(s)$. By way of contradiction, suppose $\tilde{\pi}^* \in \arg\min_{\pi^*\in\Pi^*}\{\sum_{s\in S} U\big(f'(s)\big)\pi^*(s) + U(q')\pi^*(s^*)\}$. By the above argument, $\bar{\pi}^*(s^*) = \tilde{\pi}^*(s^*)$. Hence,

$$\sum_{s\in S} U\big(f'(s)\big)\tilde{\pi}^*(s) + U(q')\tilde{\pi}^*(s^*) \leq \sum_{s\in S} U\big(f'(s)\big)\bar{\pi}^*(s) + U(q')\bar{\pi}^*(s^*);$$

is equivalent to

$$\sum_{s\in S} U\big(f'(s)\big)\frac{\tilde{\pi}^*(s)}{\sum_{s'\in S}\tilde{\pi}^*(s')} \leq \sum_{s\in S} U\big(f'(s)\big)\bar{\pi}^{**}(s);$$

However, the above inequality contradicts the choice of π^{**} since $(\frac{\tilde{\pi}^*(s)}{\sum_{s'\in S}\tilde{\pi}^*(s')})_{s\in S} \in \Pi^*|_S$.

Step 6.3. Since $\bar{\pi}^*$ is the minimizer of $\min_{\pi^*\in\Pi^*}\{\sum_{s\in S} U\big(f'(s)\big)\pi^*(s) + U(q')\pi^*(s^*)\}$ where either $f'(s) \succ_{\hat{F}} q'$ for each $s \in S$ or $q' \succ_{\hat{F}} f'(s)$ for each $s \in S$, by Step 6.1 and Equation (23),

$$\sum_{s\in S} U\big(f'(s)\big)\pi^{**}(s) \geq \min_{\pi\in\Pi}\sum_{s\in S} U\big(f'(s)\big)\pi(s).$$

However, the above inequality contradicts the choice of π^{**} since $\Pi \subset \Pi^{**}$ and $\pi^{**} \notin \Pi$.

Necessity of CEC. Take any $f \in \hat{F}$ and $p \in \Delta(C)$. Let $\bar{\pi}^* \in \arg\min_{\pi^*\in\Pi^*}\{\sum_{s\in S} U\big(f'(s)\big)\pi^*(s) + U(p)\pi^*(s^*)\}$ and $\bar{\pi}$ be the minimizer of Equation (18). The following two steps jointly prove the necessity of CEC.

Step 1. $f \sim_{\hat{F}} p$ implies $(f(s)\, E_s)_{s\in S} \cup p \succsim_{\hat{F}_1} p$.

Note that $(f(s)\, E_s)_{s\in S} \cup p \succsim_{\hat{F}_1} p$ is equivalent to $\sum_{s\in S} U\big(f(s)\big)\bar{\pi}^*(s) + U(p)\bar{\pi}^*(s^*) \geq U(p)$; equivalently,

$$\sum_{s\in S} U\big(f(s)\big)\frac{\bar{\pi}^*(s)}{\sum_{s\in S}\bar{\pi}^*(s)} \geq U(p) = \min_{\pi\in\Pi}\sum_{s\in S} U\big(f(s)\big)\pi(s).$$

The above inequality holds since $(\frac{\bar{\pi}^*(s)}{\sum_{s\in S}\bar{\pi}^*(s)})_S \in \Pi$ by reverse full-Bayesianism.

Step 2. $f \sim_{\hat{F}} p$ implies $(f(s)\, E_s)_{s\in S} \cup p \precsim_{\hat{F}_1} p$.

By reverse full-Bayesianism, there is $\pi_1 \in \Pi_1$ such that $\frac{\bar{\pi}(s)}{\bar{\pi}(s')} = \frac{\pi_1(E_s)}{\pi_1(E_{s'})}$ for all $s, s' \in S$. Since $\bar{\pi}^* \in \arg\min_{\pi^*\in\Pi^*}\{\sum_{s\in S} U\big(f'(s)\big)\pi^*(s) + U(p)\pi^*(s^*)\}$, we have

$$\sum_{s\in S} U\big(f(s)\big)\bar{\pi}^*(s) + U(p)\bar{\pi}^*(s^*) \leq \sum_{s\in S} U\big(f(s)\big)\pi_1(E_s) + U(p)(1 - \sum_{s\in S}\pi_1(E_s)).$$

It is enough to show that $\sum_{s\in S} U\big(f(s)\big)\pi_1(E_s) + U(p)(1 - \sum_{s\in S}\pi_1(E_s)) = U(p)$; equivalently,

$$\sum_{s\in S} U\big(f(s)\big)\frac{\pi_1(E_s)}{\sum_{s\in S}\pi_1(E_s)} = U(p) = \sum_{s\in S} U\big(f(s)\big)\bar{\pi}(s).$$

The above equality holds since $\frac{\pi_1(E_s)}{\sum_{s\in S}\pi_1(E_s)} = \bar{\pi}(s)$ by definition of π_1.

Necessity of NCEI. Take any $f \in \hat{F}$ and $p, q \in \Delta(C)$ such that $f \sim_{\hat{F}} p$ and either $f(s) \succ_{\hat{F}} q$ for each $s \in S$ or $q \succ_{\hat{F}} f(s)$ for each $s \in S$. Let $\bar{\pi}^* \in \arg\min_{\pi^* \in \Pi^*}\{\sum_{s \in S} U\big(f'(s)\big)\pi^*(s) + U(q)\pi^*(s^*)\}$. Similar to what we have proven in Step 2 and Step 6.1 of the sufficiency part , $(f(s)\, E_s)_{s \in S} \cup q \succsim_{\hat{F}_1} p\, E(S) q$ is equivalent to

$$\sum_{s \in S} U\big(f(s)\big) \frac{\bar{\pi}^*(s)}{\sum_{s \in S} \bar{\pi}^*(s)} \geq U(p).$$

Since $f \sim_{\hat{F}} p$ implies $\min_{\pi \in \Pi} \sum_{s \in S} U\big(f(s)\big)\pi(s) = U(p)$, we shall prove that

$$\sum_{s \in S} U\big(f(s)\big) \frac{\bar{\pi}^*(s)}{\sum_{s \in S} \bar{\pi}^*(s)} \geq \min_{\pi \in \Pi} \sum_{s \in S} U\big(f(s)\big)\pi(s).$$

By reverse full-Bayesianism, we have $(\frac{\bar{\pi}^*(s)}{\sum_{s \in S} \bar{\pi}^*(s)})_S \in \Pi$. Hence, the above inequality holds.

A.5 Proof of Proposition 1

We first state the following useful lemma.

Lemma 3. *A MEU preference $\succsim$ satisfies $\mathcal{E}$-Independence if and only if it is $\mathcal{E}$-unambiguous.*

We omit the proof of Lemma 3 since it is essentially identical to the second half of the proof of Theorem 1.

Proof of Proposition 1. By Lemma 3, $\succcurlyeq_{\hat{F}}$ satisfies $\mathcal{P}$-Independence if and only if $\succcurlyeq_{\hat{F}}$ is $\mathcal{P}$-unambiguous, and $\succcurlyeq_{\hat{F}_1}$ satisfies $\mathcal{P}_1$-Independence if and only if $\succcurlyeq_{\hat{F}_1}$ is $\mathcal{P}_1$-unambiguous. Therefore, $\succcurlyeq_{\hat{F}}$ satisfies $\mathcal{P}$-Independence and $\succcurlyeq_{\hat{F}_1}$ satisfies $\mathcal{P}_1$-Independence if and only if $\succcurlyeq_{\hat{F}}$ is $\mathcal{P}$-unambiguous and $\succcurlyeq_{\hat{F}_1}$ is a PD-unambiguity consistent extension of $\succcurlyeq_{\hat{F}}$.

A.6 Proof of Theorem 4

Lemma 4. *For any MEU preference relation $\succsim$ with (Π, U), $\succsim$ satisfies $\mathcal{E}$-Separability if and only if Π is $\mathcal{E}$-rectangular.*

Proof of Lemma 4. Take any extreme point $\bar{\pi}$ of Π and $E \in \mathcal{E}$. Take acts $f, h \in \hat{F}$ such that $\bar{\pi} \in \arg\min_{\pi \in \Pi}\{\sum_{s \in E} U(f(s))\pi(s) + \sum_{s \in E^c} U(h(s))\pi(s)\}$ and $f(s) \not\sim f(s')$, $f(s) \not\sim h(s')$, and $h(s) \not\sim h(s')$ for any $s, s' \in S$. For any $p \in \Delta(C)$, $\mathcal{E}$-Separability implies that $fEh \sim pEh$ if and only if $fEw \sim pEw$. Equivalently,

$$\min_{\pi \in \Pi}\{\sum_{s \in E} U(f(s))\pi(s) + \sum_{s \in E^c} U(h(s))\pi(s)\} = \min_{\pi \in \Pi}\{U(p)\pi(E) + \sum_{s \in E^c} U(h(s))\pi(s)\}$$

if and only if

$$\min_{\pi \in \Pi}\{\sum_{s \in E} U(f(s))\pi(s)\} = \min_{\pi \in \Pi}\{\pi(E)\}U(p).$$

We can assume that $p \not\sim h(s)$ for any $s \in E^c$. Otherwise, we take $f' = \alpha f + (1-\alpha)w$ and $p' = \alpha p + (1-\alpha)\delta_w$ instead of f and p. Note that there is $\alpha \in (0, 1)$ close to 1 such that $p' \not\sim h(s)$

for any $s \in E^c$, $\bar{\pi} \in \arg\min_{\pi\in\Pi}\{\sum_{s\in E} U(f'(s))\pi(s) + \sum_{s\in E^c} U(h(s))\pi(s)\}$, and for any $s, s' \in S$, $f'(s) \not\sim f'(s')$, $f'(s) \not\sim h(s')$, and $h(s) \not\sim h(s')$. Moreover, $f'Ew \sim p'Ew$ for any value of α.

Suppose now $p \not\sim h(s)$ for any $s \in E^c$. Let $\tilde{\pi} \in \arg\min_{\pi}\{U(p)\pi(E) + \sum_{s\in E^c} U(h(s))\pi(s)\}$ and $\pi^* \in \arg\min_{\pi\in\Pi}\{\sum_{s\in E} U(f(s))\pi(s)\}$. Then we have

$$\sum_{s\in E} U(f(s))\bar{\pi}(s) + \sum_{s\in E^c} U(h(s))\bar{\pi}(s) = U(p)\tilde{\pi}(E) + \sum_{s\in E^c} U(h(s))\tilde{\pi}(s) \tag{25}$$

if and only if

$$\sum_{s\in E} U(f(s))\pi^*(s) = \min_{\pi\in\Pi}\{\pi(E)\}U(p).$$

We first prove that $\sum_{s\in E} U(f(s))\bar{\pi}(s) = U(p)\tilde{\pi}(E)$. Note that $fEw \sim pEw$ implies that $f_\alpha Ew \sim p_\alpha Ew$ for any $\alpha \in [0,1]$ where $f_\alpha = \alpha f + (1-\alpha)w$ and $p_\alpha = \alpha p + (1-\alpha)\delta_w$. When α is close to 1, $\bar{\pi} \in \arg\min_{\pi\in\Pi}\{\sum_{s\in E} U(f_\alpha(s))\pi(s) + \sum_{s\in E^c} U(h(s))\pi(s)\}$ since $f(s) \not\sim h(s')$ for any $s, s' \in S$ and $\tilde{\pi} \in \arg\min_{\pi\in\Pi}\{U(p_\alpha)\pi(E) + \sum_{s\in E^c} U(h(s))\pi(s)\}$ since $p \not\sim h(s)$ for any $s \in E^c$. Moreover, $\pi^* \in \arg\min_{\pi\in\Pi}\{\sum_{s\in E} U(f_\alpha(s))\pi(s)\}$ for any $\alpha \in (0,1]$. Therefore, we have

$$\sum_{s\in E} U(f_\alpha(s))\bar{\pi}(s) + \sum_{s\in E^c} U(h(s))\bar{\pi}(s) = U(p)\tilde{\pi}_\alpha(E) + \sum_{s\in E^c} U(h(s))\tilde{\pi}(s).$$

Combining the above equation with Equation (25), since h is common in the both equations, we have

$$\sum_{s\in E} U(f_\alpha(s))\bar{\pi}(s) - U(p)\tilde{\pi}_\alpha(E) = \alpha\big(\sum_{s\in E} U(f(s))\bar{\pi}(s) - U(p)\tilde{\pi}(E)\big) = \sum_{s\in E} U(f(s))\bar{\pi}(s) - U(p)\tilde{\pi}(E).$$

Since $\alpha < 1$, we have $\sum_{s\in E} U(f(s))\bar{\pi}(s) = U(p)\tilde{\pi}(E)$. In other words, we just proved that

$$\frac{1}{\tilde{\pi}(E)}\sum_{s\in E} U(f(s))\bar{\pi}(s) = U(p) = \frac{1}{\min_\pi\{\pi(E)\}}\sum_{s\in E} U(f(s))\pi^*(s). \tag{26}$$

We now prove that $\frac{\pi^*(s)}{\min_\pi\{\pi(E)\}} = \frac{\bar{\pi}(s)}{\tilde{\pi}(E)}$ for any $s \in S$. Take any state $\bar{s} \in S$ and act f' such that $f'(s) = f(s)$ for any $s \neq \bar{s}$ and $|U(f'(\bar{s})) - U(f(\bar{s}))| = \epsilon > 0$. Take p' such that $f'Ew \sim p'Ew$. When ϵ is small enough, $\bar{\pi}$ is still the minimizer of $\min_\pi\{\sum_{s\in E} U(f'(s))\pi(s) + \sum_{s\in E^c} U(h(s))\pi(s)\}$ since $f(s) \not\sim h(s')$ for any $s, s' \in S$, $\tilde{\pi} \in \arg\min_\pi\{U(p')\pi(E) + \sum_{s\in E^c} U(h(s))\pi(s)\}$ since $p \not\sim h(s)$ for any $s \in E^c$, $\pi^* \in \arg\min_\pi\{\sum_{s\in E} U(f_\alpha(s))\pi(s)\}$ since $f(s) \neq f(s')$ for any $s, s' \in S$. Then by (26), we will have

$$\frac{1}{\tilde{\pi}(E)}\sum_{s\in E} U(f'(s))\bar{\pi}(s) = U(p') = \frac{1}{\min_{\pi\in\Pi}\{\pi(E)\}}\sum_{s\in E} U(f'(s))\pi^*(s). \tag{27}$$

Subtracting (26) from (27), we obtain $\frac{\bar{\pi}(s)}{\tilde{\pi}(E)}(U(f'(\bar{s})) - U(f(\bar{s}))) = \frac{\pi^*(s)}{\min_{\pi\in\Pi}\{\pi(E)\}}(U(f'(\bar{s})) - U(f(\bar{s})))$.

Hence, $\frac{\bar{\pi}(\bar{s})}{\bar{\pi}(E)} = \frac{\pi^*(\bar{s})}{\min_{\pi\in\Pi}\{\pi(E)\}}$. From (25), we also obtain

$$\sum_{s\in E^c} U(h(s))\bar{\pi}(s) = \sum_{s\in E^c} U(h(s))\tilde{\pi}(s).$$

By essentially repeating the above argument, we can prove that $\bar{\pi}(s) = \tilde{\pi}(s)$ for any $s \in E^c$. Hence, we obtain $\bar{\pi}(E) = \tilde{\pi}(E)$. In other words, we proved that $\bar{\pi}|_E = (\frac{\bar{\pi}(s)}{\sum_{s'\in E}\bar{\pi}(s')})_{s\in E} = \pi^*|_E = (\frac{\pi^*(s)}{\sum_{s'\in E}\pi^*(s')})_{s\in E}$.

Now take any $f \in \hat{F}$. For each E, take $p_E \in \Delta(C)$ such that $fEw \sim p_E Ew$; equivalently,

$$\min_{\pi\in\Pi}\{\sum_{s\in E} U(f(s))\pi(s)\} = \min_{\pi\in\Pi}\{\pi(E)\}U(p_E).$$

Notice that $f \sim p_E Ef$. By repeatedly using $\mathcal{E}$-Separability and replacing f with p_E on each E, we obtain $f \sim (p_E E)_{E\in\mathcal{E}}$. Equivalently, by choices of p_E,

$$\min_{\pi\in\Pi}\{\sum_{s\in S} U(f(s))\pi(s)\} = \min_{\pi\in\Pi}\{\sum_{E\in\mathcal{E}} \pi(E)U(p_E)\} = \min_{\pi\in\Pi}\{\sum_{E\in\mathcal{E}} \pi(E)\frac{\min_{\pi\in\Pi}\{\sum_{s\in E} U(f(s))\pi(s)\}}{\min_{\pi\in\Pi}\{\pi(E)\}}\}.$$

Since we proved that $\pi^*(E) = \min_{\pi\in\Pi}\{\pi(E)\}$ where $\pi^* \in \arg\min_{\pi\in\Pi}\{\sum_{s\in E} U(f(s))\pi(s)\}$, the above equality implies

$$\min_{\pi\in\Pi}\{\sum_{s\in S} U(f(s))\pi(s)\} = \min_{\pi\in\Pi}\{\sum_{E\in\mathcal{E}} \pi(E)\min_{\pi\in\Pi}\{\sum_{s\in E} U(f(s))\frac{\pi(s)}{\pi(E)}\}\}.$$

Equivalently,

$$\min_{\pi\in\Pi}\{\sum_{s\in S} U(f(s))\pi(s)\} = \min_{\pi\in\Pi}\Big\{\sum_{E\in\mathcal{E}} \pi(E)\min_{\pi|_E\in\Pi|_E}\Big\{\sum_{s\in E} U(f(s))\pi|_E(s)\Big\}\Big\}.$$

Since we proved that $\bar{\pi}|_E = \pi^*|_E$ where $\bar{\pi}$ was an arbitrarily chosen extreme point of Π such that $\bar{\pi} \in \arg\min_{\pi\in\Pi}\{\sum_{s\in E} U(f(s))\pi(s) + \sum_{s\in E^c} U(h(s))\pi(s)\}$ and $\pi|_E$ is decided before the choice of π for the outside min operator, we have

$$\Pi = \{\sum_{E\in\mathcal{E}} \pi(E)\,\pi|_E : \text{ for some } \pi \in \Pi \text{ and } \pi|_E \in \Pi|_E \text{ for each } E \in \mathcal{E}\}.$$

Proof of Theorem 4. By Lemma 4, $\succsim_{\hat{F}}$ satisfies $\mathcal{P}$-Separability if and only if Π is $\mathcal{P}$-rectangular and $\succsim_{\hat{F}_1}$ satisfies $\mathcal{P}_1$-Separability if and only if Π_1 is $\mathcal{P}_1$-rectangular. Moreover, by Theorem 3, $\succsim_{\hat{F}}$ and $\succsim_{\hat{F}_1}$ jointly satisfy CEC and NCEI if and only if $\succsim_{\hat{F}}$ is a reverse full-Bayesian extension of $\succsim_{\hat{F}_1}$ and there are α, β with $\alpha > 0$ such that $U_1(p) = \alpha U(p) + \beta$ for any $p \in \Delta(C)$. Therefore, (i) and (ii) are equivalent.

A.7 Proof of Proposition 2

We first prove the following useful lemmas.

Lemma 5. *For any MEU preference $\succsim$ with (Π, U), $\succsim$ is $\mathcal{E}$-unambiguous and Π is $\mathcal{E}$-rectangular if and only if $\succsim$ is $\mathcal{E}$-additive-separable.*

Proof of Lemma 5. It is straightforward that $\mathcal{E}$-additive-separability implies $\mathcal{E}$-unambiguity. Therefore, it is enough to prove that under $\mathcal{E}$-unambiguity, $\mathcal{E}$-rectangularity and $\mathcal{E}$-additive-separability are equivalent. Take any MEU preference $\succcurlyeq$ with (Π, U) that is $\mathcal{E}$-unambiguous. Let $\pi^*(E) = \pi(E)$ for any $\pi \in \Pi$. Then Π is $\mathcal{E}$-rectangular if and only if

$$\Pi = \Big\{ \sum_{E \in \mathcal{E}} \pi^*_E \cdot \pi_E \Big| \pi_E \in \Pi_E \text{ for each } E \in \mathcal{E} \Big\} = \sum_{E \in \mathcal{E}} \pi^*_E \, \Pi_E.$$

In other words, Π is $\mathcal{E}$-rectangular if and only if

$$\min_{\pi \in \Pi} \sum_{s \in S} U\big(f(s)\big)\pi(s) = \sum_{E \in \mathcal{E}} \min_{\pi \in \Pi_E} \Big\{ \sum_{s \in E} U\big(f(s)\big)\pi(s) \Big\} \text{ for all } f \in \hat{F}.$$

Lemma 6. *A MEU preference $\succsim$ satisfies $\mathcal{E}$-Separability and $\mathcal{E}$-Independence if and only if $\succsim$ is $\mathcal{E}$-additive-separable.*

Proof of Lemma 6. By Lemma 3, $\mathcal{E}$-unambiguity is equivalent to $\mathcal{E}$-Independence. By Lemma 4, $\mathcal{E}$-rectangularity is equivalent to $\mathcal{E}$-Separability. Therefore, $\succsim$ satisfies $\mathcal{E}$-Separability and $\mathcal{E}$-Independence if and only if $\succsim$ is $\mathcal{E}$-unambiguous and Π is $\mathcal{E}$-rectangular. Finally, by Lemma 5, $\succsim$ satisfies $\mathcal{E}$-Separability and $\mathcal{E}$-Independence if and only if $\succsim$ is $\mathcal{E}$-additive-separable.

Proof of Proposition 2. The equivalence between (*i*) and (*iii*) follows from Lemma 6. The equivalence between (*ii*) and (*iii*) follows from Lemma 5. Therefore, (*i*), (*ii*), and (*iii*) are equivalent.

A.8 Proofs of Corollaries 2-4

Proof of Corollary 2. As shown in Theorem 2, $\pi(E_s) = \pi(E(S))\mu(s)$ for any $s \in S$ and $\pi \in \Pi_1$. Since $\pi(E(S)) = \pi(S_1) = 1$ in the context of discovering acts, $\pi(E_s) = \mu(s)$ for any $s \in S$ and $\pi \in \Pi_1$. Therefore, Unambiguity Consistency is satisfied.

Proof of Corollary 3. As shown in Theorem 2, $\pi(E_s) = \pi(E(S))\mu(s)$ for any $s \in S$ and $\pi \in \Pi_1$. Let $\bar{\delta} = \max_{\pi \in \Pi_1}\{\pi(E(S))\}$ and $\underline{\delta} = \min_{\pi \in \Pi_1}\{\pi(E(S))\}$. Since $E_s = s^1$ for some $s^1 \in S_1$, we obtain $\pi(s^1) = \delta\mu(s)$ where $\delta \in [\underline{\delta}, \bar{\delta}]$.

Proof of Corollary 4. By Corollary 2, Unambiguity Consistency is satisfied. Hence, $\succsim_{\hat{F}_1}$ is $\mathcal{P}_1$-unambiguous. Then by Lemma 5, $\mathcal{P}_1$-rectangularity is equivalent to $\mathcal{P}_1$-additive-separability.

B Parametric MEU: Axiomatic Characterization

In this section, we axiomatically characterize the parametric MEU introduced in Section 6. To characterize the representation (13), we need to strengthen NUI as follows.

(A.14) (**Extreme NUI**) For all $f, g \in \hat{F}_1$, $q \in \Delta(C_1)$ and $\alpha \in [0, 1]$,

$$\text{if } \; f \succcurlyeq_{\hat{F}_1} q, \;\; \text{then} \;\; \alpha f + (1-\alpha)g \succcurlyeq_{\hat{F}_1} \alpha q + (1-\alpha)g,$$

and if for each $s \in S$, there is $s^1 \in E_s$ such that $f(\tilde{s}^1) \succcurlyeq_{\hat{F}_1} f(s^1)$ and $g(\tilde{s}^1) \succcurlyeq_{\hat{F}_1} g(s^1)$ for all $\tilde{s}^1 \in E_s$, then

$$f \sim_{\hat{F}_1} q \quad \text{if and only if} \quad \alpha f + (1-\alpha)g \sim_{\hat{F}_1} \alpha q + (1-\alpha)g.$$

Similar to NUI, the first part of Extreme NUI suggests that the objective lottery q suffers more (or gains less) than the subjective act f from mixtures that eliminate its objective appeal. However, the second part suggests that the objective lottery q and the subjective act f equally suffer (or equally gain) when they are mixed with an act that agrees with f on the worst state in E_s.[35]

Under Extreme NUI, the extended MEU representation satisfies both Unambiguity Consistency and Likelihood Consistency.

Theorem 5. *Suppose that $S_1 = S_1^R$. Then, the following two statements are equivalent:*

(i) $\succcurlyeq_{\hat{F}}$ satisfies axioms (A.1)-(A.5), $\succcurlyeq_{\hat{F}_1}$ satisfies axioms (A.1)-(A.4), and Extreme NUI, and $\succcurlyeq_{\hat{F}}$ and $\succcurlyeq_{\hat{F}_1}$ jointly satisfy BAC.

(ii) There exist a non-constant and affine function $U : \Delta(C) \to \mathbb{R}$, a probability measure $\mu \in \Delta(S)$, and a set $\{(\eta_s, \alpha_s)\}_{s\in S} \in \prod_{s\in S} (\Delta(E_s) \times [0, 1])$ such that $\succcurlyeq_{\hat{F}}$ is a SEU preference with (μ, U) and for all $f, g, \in \hat{F}_1$, $f \succcurlyeq_{\hat{F}_1} g$ if and only if

$$\sum_{s\in S} \mu(s)\Big(\min_{\pi\in\Pi_{E_s}^{(\eta_s,\alpha_s)}} \sum_{s^1\in E_s} U\big(f(s^1)\big)\pi(s^1)\Big) \geqslant \sum_{s\in S} \mu(s)\Big(\min_{\pi\in\Pi_{E_s}^{(\eta_s,\alpha_s)}} \sum_{s^1\in E_s} U\big(g(s^1)\big)\pi(s^1)\Big), \tag{28}$$

where

$$\Pi_{E_s}^{(\eta_s,\, \alpha_s)} = \alpha_s\{\eta_s\} + (1-\alpha_s)\Delta(E_s).$$

Consistent with Corollary 2, Theorem 5 (specially, Equation (28)) implies that $\succcurlyeq_{\hat{F}_1}$ is an unambiguity and likelihood consistent extension of the original preference $\succcurlyeq_{\hat{F}}$. Moreover, the DM's new beliefs Π_1 on the expanded state space S_1 take the following form:

$$\Pi_1 = \times_{s\in S}\, \mu(s)\, \Pi_{E_s}^{(\eta_s,\, \alpha_s)} = \times_{s\in S} \Big\{\mu(s)\, \alpha_s\{\eta_s\} + \mu(s)(1-\alpha_s)\Delta(E_s)\Big\}. \tag{29}$$

[35] Recall that NUI requires that bets $g = p_{E_s} r$ cannot be used to hedge against ambiguity. Notice that for any act f and a bet $g = p_{E_s} r$, f and g satisfy the following property in Extreme NUI: for any $s \in S$, there exists $s^1 \in E_s$ with $f(\tilde{s}^1) \succcurlyeq f(s^1)$, $g(\tilde{s}^1) \succcurlyeq g(s^1)$ for all $\tilde{s}^1 \in E_s$. Hence, Extreme NUI is stronger than NUI. The second part of Extreme NUI is adopted from Eichberger and Kelsey (1999).

The proof of Theorem 5 is in the online appendix.

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Online Appendix: Ambiguity under Growing Awareness

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Appendix D: Proof of Theorem 5

We only prove the sufficiency part. Since Extreme NUI is stronger than NUI, by Theorem 2, there are (μ, U) and (Π_1, U) such that $\succcurlyeq_{\hat{F}}$ admits a SEU representation with (μ, U) and $\succcurlyeq_{\hat{F}_1}$ admits a MEU representation with (Π_1, U). Moreover, $\succcurlyeq_{\hat{F}_1}$ is an unambiguity consistent and likelihood consistent extension of $\succcurlyeq_{\hat{F}}$. That is, $\pi(E_s) = \mu(s)$ for any $s \in S$ and $\pi \in \Pi_1$. Without loss of generality, let $U(w) = 0$ where w is the worst consequence in C. We prove the theorem in three steps.

Step 1: For any $f, h \in \hat{F}_1$ and $s \in S$,

$$\min_{\pi\in\Pi_1}\Big\{\sum_{s\in E_s} U(f(s)\pi(s) + \sum_{s\in S_1\setminus E_s} U(h(s))\pi(s)\Big\} = \min_{\pi\in\Pi_1}\sum_{s\in E_s} U(f(s)\pi(s) + \min_{\pi\in\Pi_1}\sum_{s\in S_1\setminus E_s} U(h(s))\pi(s).$$

Let us fix $f, h \in \hat{F}_1$ and $s \in S$. Take any $p \in \Delta(C)$ such that $fE_s w \sim_{\hat{F}_1} pE_s w$. Notice that $fE_s w$ and $pE_s h$ as well as $pE_s w$ and $pE_s h$ agree on the worst state of each event in $\{E_{s'}\}_{s'\in S}$. Therefore, by Extreme NUI, for any $\alpha \in [0, 1]$,

$$(27) \qquad (\alpha f + (1-\alpha)p)E_s(\alpha w + (1-\alpha)h) \sim_{\hat{F}_1} pE_s(\alpha w + (1-\alpha)h);$$

equivalently,

$$(28) \qquad V^{MEU}\big(\underbrace{(\alpha f + (1-\alpha)p)E_s(\alpha w + (1-\alpha)h)}_{g}\big) = V^{MEU}\big(\underbrace{pE_s(\alpha w + (1-\alpha)h)}_{\tilde{g}}\big).$$

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Since E_s is unambiguous and $U(w)=0$,

$$V^{MEU}(g) = \min_{\pi\in\Pi_1}\Big\{\alpha\sum_{s\in E_s}U(f(s))\pi(s) + (1-\alpha)\pi(E_s)U(p) + (1-\alpha)\sum_{s\in S_1\setminus E_s}U(h(s))\pi(s)\Big\}$$
$$=(1-\alpha)U(p)\pi(E_s) + \min_{\pi\in\Pi_1}\Big\{\alpha\sum_{s\in E_s}U(f(s))\pi(s) + (1-\alpha)\sum_{s\in S_1\setminus E_s}U(h(s))\pi(s)\Big\}$$

and

$$V^{MEU}(\tilde{g}) = \min_{\pi\in\Pi_1}\Big\{U(p)\pi(E_s)+(1-\alpha)\sum_{s\in S_1\setminus E_s}U(h(s))\pi(s)\Big\} = U(p)\pi(E_s)+(1-\alpha)\min_{\pi\in\Pi_1}\Big\{\sum_{s\in S_1\setminus E_s}U(h(s))\pi(s)\Big\}.$$

Thus, when $\alpha=\frac{1}{2}$, we get $V^{MEU}(g)=V^{MEU}(\tilde{g})$ if and only if

$$\min_{\pi\in\Pi_1}\Big\{\sum_{s\in E_s}U(f(s)\pi(s) + \sum_{s\in E_s}U(h(s))\pi(s)\Big\} = U(p)\pi(E_s) + \min_{\pi\in\Pi_1}\sum_{s\in E_s}U(h(s))\pi(s).$$

Moreover, since $fE_sw \sim_{\hat{F}_1} pE_sw$ and $U(w)=0$, we have $\min_{\pi\in\Pi_1}\sum_{s\in E_s}U(f(s)\pi(s) = U(p)\pi(E_s)$.

By combining the last equalities, we obtain

$$\min_{\pi\in\Pi_1}\Big\{\sum_{s\in E_s}U(f(s)\pi(s) + \sum_{s\in S_1\setminus E_s}U(h(s))\pi(s)\Big\} = \min_{\pi\in\Pi_1}\sum_{s\in E_s}U(f(s)\pi(s) + \min_{\pi\in\Pi_1}\sum_{s\in S_1\setminus E_s}U(h(s))\pi(s).$$

Step 1 essentially proves the extended MEU preference $\succsim_{\hat{F}_1}$ is additively separable across the events $\{E_s\}_{s\in S}$. That is,

$$\min_{\pi\in\Pi_1}\left\{\sum_{s\in S}\mu(s)\sum_{s_l^1\in E_s}U(h(s_l^1))\frac{\pi(s_l^1)}{\pi(E_s)}\right\}$$

is minimized at each event E_s, separately. Therefore, we obtain the following representation:

$$V^{MEU}(f) = \sum_{s\in S}\mu(s)\min_{\pi_s\in\Pi_s^{F_1}}\left\{\sum_{s_k^1\in E_s}\pi_s(s_k^1)U\big(f(s_k^1)\big)\right\}, \tag{29}$$

where $\Pi_s^{F_1}\subseteq\Delta(E_s)$. In next steps, we prove that beliefs on E_s take the form $\Pi_s^{F_1}=\beta_s\{\eta_s\}+(1-\beta_s)\,\Delta(E_s)$ for some $\beta_s\in[0,1]$ and $\eta_s\in\Delta(E_s)$.

Fix an event E_s. Take any $s_t^1\in E_s$. Let us take acts $f,g,h\in\hat{F}_1$ such that for any $s_j^1\in E_s$, $f(s_j^1)\succsim_{\hat{F}_1} f(s_t^1)$, $g(s_j^1)\succsim_{\hat{F}_1} g(s_t^1)$, and $h(s_j^1)\succsim_{\hat{F}_1} h(s_t^1)$, and $f(s^1)=g(s^1)=h(s^1)=w$ for any $s^1\in S_1\setminus E_s$. Then, by Extreme NUI, for any $\alpha\in(0,1]$,

$$f\sim_{\hat{F}_1} g \text{ iff } \alpha f+(1-\alpha)h \sim_{\hat{F}_1} \alpha g+(1-\alpha)h;$$

equivalently,

$$(30)\qquad \min_{\pi_s\in\Pi_s^{F_1}}\left\{\sum_{s_k^1\in E_s}\pi_s(s_k^1)U\big(f(s_k^1)\big)\right\}=\min_{\pi_s\in\Pi_s^{F_1}}\left\{\sum_{s_k^1\in E_s}\pi_s(s_k^1)U\big(g(s_k^1)\big)\right\}\text{ iff}$$

$$\min_{\pi_s\in\Pi_s^{F_1}}\left\{\sum_{s_k^1\in E_s}\pi_s(s_k)U\big(\alpha f(s_k^1)+(1-\alpha)h(s_k^1)\big)\right\}=\min_{\pi_s\in\Pi_s^{F_1}}\left\{\sum_{s_k^1\in E_s}\pi_s(s_k^1)U\big(\alpha g(s_k^1)+(1-\alpha)h(s_k^1)\big)\right\}.$$

Step 2: Let $f(s_t^1)=g(s_t^1)=h(s_t^1)=w$. Then (30) is equivalent to

$$\min_{\pi_s\in\Pi_s^{F_1}}\left\{\sum_{s_k^1\in E_s\setminus s_t^1}\pi_s(s_k^1)U\big(f(s_k^1)\big)\right\}=\min_{\pi_s\in\Pi_s^{F_1}}\left\{\sum_{s_k^1\in E_s\setminus s_t^1}\pi_s(s_k^1)U\big(g(s_k^1)\big)\right\}\text{ iff}$$

$$\min_{\pi_s\in\Pi_s^{F_1}}\left\{\sum_{s_k^1\in E_s\setminus s_t^1}\pi_s(s_k)U\big(\alpha f(s_k^1)+(1-\alpha)h(s_k^1)\big)\right\}=\min_{\pi_s\in\Pi_s^{F_1}}\left\{\sum_{s_k^1\in E_s\setminus s_t^1}\pi_s(s_k^1)U\big(\alpha g(s_k^1)+(1-\alpha)h(s_k^1)\big)\right\}.$$

The above equivalence is in fact the Independence Axiom on $E_s\setminus s_t^1$. Therefore, there is $\overline{\pi}_s^t\in\Delta(E_s\setminus s_t^1)$ such that

$$\min_{\pi_s\in\Pi_s^{F_1}}\left\{\sum_{s_k^1\in E_s\setminus s_t^1}\pi_s(s_k^1)U\big(f(s_k^1)\big)\right\}=\sum_{s_k^1\in E_s\setminus s_t^1}\overline{\pi}_s^t(s_k^1)U\big(f(s_k^1)\big).$$

For any $s_j^1\in E_s\setminus s_t^1$, let $f(s_k^1)=w$ for any $s_k^1\neq s_j^1$. Then the above equation implies that $\min_{\pi_s\in\Pi_s^{F_1}}\left\{\pi_s(s_j^1)\right\}=\overline{\pi}_s^t(s_j^1)$. Since $\min_{\pi_s\in\Pi_s^{F_1}}\left\{\pi_s(s_j^1)\right\}$ is independent of s_t^1, we shall write $\overline{\pi}_s(s_j^1)$ instead of $\overline{\pi}_s^t(s_j^1)$.

Let $\pi_s^*(s_t^1)\equiv\max_{\pi_s\in\Pi_s^{F_1}}\{\pi_s(s_t^1)\}$.

Step 3: Suppose $s_j^1\in E_s\setminus s_t^1$, $f(s_j^1)\succ_{\hat{F}_1}f(s_t^1)$, $g(s_j^1)\succ_{\hat{F}_1}g(s_t^1)$, and $h(s_j^1)\succ_{\hat{F}_1}h(s_t^1)$, and $f(s_t^1)=g(s_t^1)=h(s_t^1)$. Moreover, suppose $f(s_j^1)=f(s_k^1)$ for any $s_j^1,s_k^1\neq s_t^1$.

Suppose $U(f(s_t^1))=U(g(s_t^1))=U(h(s_t^1))$ is small enough relative to $U(f(s_k^1))$, $U(g(s_k^1))$, and $U(g(s_k^1))$. Then (30) is equivalent to

$$\pi_s^*(s_t^1)U(f(s_t^1))+(1-\pi_s^*(s_t^1))U(f(s_k^1))=\big(1-\sum_{s_k^1\in E_s\setminus s_t^1}\overline{\pi}_s(s_k^1)\big)U(g(s_t^1))+\sum_{s_k^1\in E_s\setminus s_t^1}\overline{\pi}_s(s_k^1)U\big(g(s_k^1)\big)$$

if and only if

$$(1-\sum_{s_k^1\in E_s\setminus s_t^1}\overline{\pi}_s(s_k^1))U\big(\alpha f(s_t^1)+(1-\alpha)h(s_t^1)\big)+\sum_{s_k^1\in E_s\setminus s_t^1}\overline{\pi}_s(s_k^1)U\big(\alpha f(s_k^1)+(1-\alpha)h(s_k^1)\big)$$

$$= (1 - \sum_{s_k^1 \in E_s \setminus s_t^1} \overline{\pi}_s(s_k^1)) U\big(\alpha g(s_t^1) + (1-\alpha) h(s_t^1)\big) + \sum_{s_k^1 \in E_s \setminus s_t^1} \overline{\pi}_s(s_k^1) U\big(\alpha g(s_k^1) + (1-\alpha) h(s_k^1)\big).$$

The above equivalence implies $\pi_s^*(s_t^1) = 1 - \sum_{s_k^1 \in E_s \setminus s_t^1} \overline{\pi}_s(s_k^1)$. Similarly, we have $\pi_s^*(s_{t'}^1) = 1 - \sum_{s_k^1 \in E_s \setminus s_{t'}^1} \overline{\pi}_s(s_k^1)$. These two equalities imply that $\pi_s^*(s_t^1) - \overline{\pi}_s(s_t^1) = \pi_s^*(s_{t'}^1) - \overline{\pi}_s(s_{t'}^1)$.

Since $\pi_s^*(s_t^1) \geq \overline{\pi}_s(s_t^1)$, let $1 - \alpha_s \equiv \pi_s^*(s_t^1) - \overline{\pi}_s(s_t^1)$. Finally, since $\pi_s^*(s_t^1) + \sum_{s_k^1 \neq s_t^1} \overline{\pi}_s(s_k^1) = 1$, we have $\sum_{s_k^1 \in E_s} \overline{\pi}_s(s_k^1) = \alpha_s$. Let

$$\eta_s(s_t^1) \equiv \frac{\overline{\pi}_s(s_t^1)}{\sum_{s_k^1 \in E_s} \overline{\pi}_s(s_k^1)} = \frac{\overline{\pi}_s(s_t^1)}{\alpha_s}.$$

Then we have $\Pi_s^{F_1} = \{\overline{\pi}_s\} + (1 - \alpha_s)\, \Delta(E_s) = \alpha_s \, \{\eta_s\} + (1 - \alpha_s)\, \Delta(E_s)$ where $\eta_s \in \Delta(E_s)$.