# Common Belief in Choquet Rationality with an "Attitude"*

Adam Dominiak[†] Burkhard C. Schipper[‡]

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**Abstract**

We consider finite games in strategic form with Choquet expected utility. We characterize a preference-based notion of belief, define Choquet rationalizability, and characterize it by Choquet rationality and common beliefs in Choquet rationality in the universal capacity type space in a purely measurable setting. We also show that Choquet rationalizability is equivalent to iterative elimination of strictly dominated actions (not in the original game but) in an extended game. This allows for computation of Choquet rationalizable actions without the need to first compute Choquet integrals. Choquet expected utility enables us to investigate common belief in ambiguity love/aversion. We show that Choquet rationality and common belief in Choquet rationality *and* ambiguity love/aversion leads to smaller/larger sets of action profiles, respectively.

**Keywords:** Choquet expected utility, common belief, rationalizability, ambiguity aversion, ambiguity love, game theory, interactive epistemology, Knightian uncertainty.

**JEL-Classifications:** C72, D72.

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[†]Department of Economics, Virginia Tech. Email: dominiak@vt.edu

[‡]Department of Economics, University of California, Davis. Email: bcschipper@ucdavis.edu

# 1 Introduction

Choquet expected utility theory was probably the first approach to decision making under ambiguity (Schmeidler, 1986, 1989). It has been applied to a variety settings including portfolio choice (Dow and Werlang, 1992), auctions (Salo and Weber, 1995), arbitrage pricing (Kelsey and Milne, 1995), incomplete contracts (Mukerji, 1998), risk sharing (Chateauneuf, Dana, Tallon, 2000), insurance contract (Jeleva, 2000), incomplete markets (Mukerji and Tallon, 2001), public goods (Eichberger and Kelsey, 2002), search (Nishimura and Ozaki, 2004), wages (Mukerji and Tallon, 2004), peace-making (Eichberger, Kelsey, Schipper, 2009), Cournot and Bertrand oligopoly (Eichberger, Kelsey, Schipper, 2009), trade (Kajii and Ui, 2006, Dominiak, Eichberger, Lefort, 2012), agreement theorems (Dominiak and Lefort, 2013, 2015) etc. Compared to some other approaches, it is flexible enough to allow for modelling of both ambiguity aversion and ambiguity love. Moreover, since it does not feature probability measures, it forces us to conceptually reconsider standard notions of game theory that were historically developed with probability measures in mind.

Applying Choquet expected utility to games is not new. Dow and Werlang (1994), Eichberger and Kelsey (2000, 2014), Marinacci (2000), Haller (2000), Eichberger, Kelsey, Schipper (2009), Dominiak and Eichberger (2019) apply Choquet expected utility of Schmeidler (1989) to games.[1] While these papers extend formal definitions of Nash equilibrium to games with Choquet expected utility, it is less clear that also the interpretations of Nash equilibrium extend to games under ambiguity. For instance, how to interpret independence of conjectures over opponents' play? And how can mutual belief (under ambiguity) of conjectures be learned when learning under ambiguity is itself a conceptually difficult problem. Our approach is to focus on extending rationalizability à la Spohn (1982), Bernheim (1984) and Pearce (1984) to games with ambiguity and characterize it by common belief in Choquet rationality. That is, we avoid the issue of independence of conjectures by allowing for "correlated" conjectures (in particular, whether or not players are correlated may be a source of ambiguity in games). Moreover, we assume mutual belief in Choquet rationality rather than mutual belief in play.

Applying rationalizability notions to games with preferences that allow for ambiguity is also not new. In a truly seminal paper, Epstein (1997) introduced a general utility representation-based notion of rationalizability, that applies to various decision theories including essentially[2] Choquet expected utility. Although this important paper has been around since at least 1997, we could not find any application of it. Perhaps one reason is that a rationalizability notion featuring the utility representation may be of limited accessibility to applied game theorist familiar with rationalizability à la Bernheim (1984)

[1]Klibanoff (1996), Lo (1996, 1999), Aryal and Stauber (2014), Riedel and Sass (2014) apply the maxmin expected utility of Gilboa and Schmeidler (1989). Battigalli et al. (2016), Hanany et al. (2019) use the smooth-model (Klibanoff, Marinacci, and Mukerji (2005).

[2]We are not aware that the version of Choquet expected utility applied by Epstein (1997) had been already developed in 1997.

and Pearce (1984) and with iterated elimination of strictly dominated actions. That's why we introduce the analogues of rationalizability and iterated elimination of strictly dominated actions for Choquet expected utility. Altogether we define six "versions" of Choquet rationalizability and show their equivalence. This allows applied game theorist to choose the version they find most easy to work with and at the same time refer to the interpretations of other versions.

The interpretation of Choquet rationalizability is made transparent with an epistemic characterization by common belief in Choquet rationality. That is, we introduce a capacity type space that then allows us to formalize the set of types who are consistent with $k$-level belief in Choquet rationality and common belief in Choquet rationality. In order to characterize Choquet rationality by common belief in Choquet rationality, we first require a "rich" capacity type space. We apply results by Ganguli, Heifetz, and Lee (2016) to claim the existence of a Choquet expected utility representation type space. Next we claim the existence of a universal capacity type space by defining a fully faithful functor between the category of Choquet expected utility representation type spaces and the category of capacity type spaces. This is outlined in Section 3.1 and the appendix.

We should clarify upfront that we assume that players *do not* randomize between their pure actions. Besides the classical arguments against mixed actions (see the debate in Osborne and Rubinstein, 1995, Chapter 3.2), there are other arguments for not considering "mixed extensions" in our context. In choice situations under ambiguity, there are different views on how a decision maker's preference for mixtures between actions and her attitudes towards ambiguity are related. In particular, depending on how randomization is captured, an ambiguity averse subject may have a (strong) preference for mixtures between actions or be indifferent to them. Preference for randomization appears in the Anscombe-Aumann framework to Choquet expected utility (Schmeidler, 1989), where mixtures are embedded via a consequence space which is a set of (objective) lotteries over outcomes. In this setup, ambiguity aversion (which is equivalent to convexity of a capacity) is defined as a preference for mixtures between two actions among which the decision maker is indifferent (see Schmeidler, 1989). Although this is the prevalent way to interpret ambiguity aversion in the context of individual decision problems, this view is problematic for interactive situations. The critical point is the resolution of uncertainty. In the Anscombe-Aumann setup, uncertainty is resolved in two stages. In the first stage, an ambiguous state is realized. In the second stage, a lottery is played to determine the final outcome. In games with mixed actions, however, a lottery (or mixed action) is defined over player's own actions. In the first stage, a lottery is resolved to determine the player's action to be played. In the second stage, the action profile determines the final payoff. (Here we take the opponents' pure action profiles as "states of nature" in the Anscombe-Aumann setting.) In this setup, it can be shown that indifference between two actions implies indifference to any mixture thereof. That is, while the timing of the resolution of uncertainty does not matter under subjective expected utility a la Savage, it does matter under different versions of Choquet expected utility. Moreover, in the version of Choquet expected utility that fits best to games in strategic form (which is not the Anscombe-Aumann setting), players who are indifferent between two actions would also be indifferent to mixtures thereof.

We should also remark that players' (Bernoulli) utility functions (i.e., risk attitudes) are fixed throughout our analysis. In line with the standard practice in game theory, we assume that players' payoffs are measured in "utils." Hence, actions assign utility numbers to each action profile of the opponent players via outcome functions. Our main goal is to inquire into how nonadditive beliefs (i.e., strategic ambiguity) and different attitudes towards ambiguity may affect the set of rationalizable actions. To disentangle the effects of ambiguity on rationalizability from those by risk attitudes, we assume that players' utilities are known.[3] Battigalli et al. (2016) provides an elegant analysis of how ambiguity aversion and risk aversion[4], respectively, affect the set of rationalizable (justifiable) actions in the family of smooth-ambiguity preferences of Klibanoff, Marinacci and Mukerji (2005). By deriving and using a generalized version of the duality lemma of Wald (1949) and Pearce (1984), the authors show that more ambiguity aversion, as well as more risk aversion, expands the set of rationalizable actions.

The paper is organized as follows: In the next section, we introduced Choquet expected utility theory. This is followed by Section 3 by definitions of Choquet rationalizability and proof of their equivalence. Section 4 focuses on the epistemic characterization. In Section 5.1 we consider restrictions to ambiguity aversion or ambiguity love. Finally, in Section 6 we explore restrictions to ambiguity attitudes under genuine ambiguity aversion (excluding completely the case subjective expected utility). All proofs and additional material are collected in an appendix.

# 2 Decision Theoretic Preliminaries

## 2.1 Choquet Expected Utility Theory

Let $\langle \Omega, \Sigma_\Omega \rangle$ be a measurable space $\Omega$ endowed with a $\sigma$-algebra $\Sigma_\Omega$. An element $\omega \in \Omega$ is called state; an element $E \in \Sigma_\Omega$ is called event.

**Definition 1 (Capacity)** *A* capacity *on $\Sigma_\Omega$ is a set-function $\nu : \Sigma_\Omega \to \mathbb{R}$ that satisfies*

[3] We may imaging that player's utility functions are elicited in another choice situation under uncertainty (unknown probabilities) by using standard techniques known in the economic literature. To maintain consistency with our setup, one may consider elicitation techniques that do not rely on randomization device, e.g., the tradoff-method by Wakker and Deneffe (1996) or "subjective mixture" introduced by Nakamura (1990) and Gul (1990). For instance, Ghirardato and Marinacci (2004) applies the latter method to derive utility function for the general family of biseparable preferences that includes Choquet expected utility preferences.

[4] Recently, Weinstein (2016) showed that the set of rationalizable actions is not invariant to monotone and nonlinear transformations of players' utility function under subjective expected utility.

*(i) Normalization:* $\nu(\emptyset) = 0$, $\nu(\Omega) = 1$, *and*

*(ii) Monotonicity: For all* $E, F \in \Sigma_\Omega$, $E \subseteq F$ *implies* $\nu(E) \leq \nu(F)$.

Let $X$ be a set of outcomes, a subset of $\mathbb{R}$. An act is a $\Sigma_\Omega$-measurable map $f : \Omega \to X$ (i.e., $f^{-1}(x) \in \Sigma_\Omega$ for all $x \in X$). An act is simple if it can take only finitely many values. Throughout the paper, we assume that any act is simple without extra saying so.. This is w.l.o.g. since we focus on finite games. We denote a simple act by $f = (E_1, x_1; \ldots; E_n, x_n)$ where $E_1, \ldots, E_n$ is a finite partition of $\Omega$ such that $f(\omega) = x_i$ for all $\omega \in E_i$ and $i = 1, \ldots, n$. We slightly abuse our notation and write $f(E_i)$ instead of "$f(\omega) = x_i$ for all $\omega \in E_i$". A constant act yields the same outcome in all states (i.e., $f^{-1}(x) = \Omega$ for all $x \in X$). We denote by $\mathcal{F}$ the set of all (simple) $\Sigma_\Omega$-measurable acts. For any event $E \in \Sigma_\Omega$ and acts $f, g \in \mathcal{F}$, $f_E g$ denotes the (composite) act defined by

$$f_E g(\omega) = \begin{cases} f(\omega) & \text{if } \omega \in E, \\ g(\omega) & \text{otherwise.} \end{cases}$$

We denote by $\succsim$ a preference relation on $\mathcal{F}$; $\succ$ and $\sim$ are the asymmetric and symmetric parts of $\succsim$, respectively.

Let $u : X \to \mathbb{R}$ be a utility function, ranking constant acts. The Choquet expected utility of an act $f$ with respect to $u$ and $\nu$ is calculated via the Choquet integral (see Choquet, 1954). More precisely, the Choquet integral of any act $f = (E_1, x_1; \ldots; E_n, x_n)$ such that $f(E_1) \succsim f(E_2) \succsim \ldots \succsim f(E_n)$ is given by

$$\int_\Omega u(f) d\nu = \sum_{i=1}^{n} u(x_i) \big[\nu\big(\bigcup_{j=1}^{i} E_j\big) - \nu\big(\bigcup_{j=1}^{i} E_{j-1}\big)\big], \tag{1}$$

with convention that $E_0 = \emptyset$.

We assume that each $\succsim$ on $\mathcal{F}$ has a Choquet expected utility representation. Formally,

**Definition 2 (Choquet Expected Utility)** *A preference* $\succsim$ *on* $\mathcal{F}$ *admits a* Choquet expected utility *representation if there exist a utility function* $u : X \longrightarrow \mathbb{R}$ *and a capacity* $\nu : \Sigma_\Omega \longrightarrow \mathbb{R}$ *such that for all acts* $f, g \in \mathcal{F}$*:*

$$f \succsim g \quad \text{if and only if} \quad \int_\Omega u(f) d\nu \geq \int_\Omega u(g) d\nu. \tag{2}$$

*Moreover,* $\nu$ *is unique and* $u$ *is unique up to a positive affine transformation.*

The family of Choquet expected utility preferences has been characterized in terms of properties on preferences in various decision theoretic settings. Such axiomatic foundations in a subjective Savage-style setup consistent with ours have been provided by

Köbbeling and Wakker (2003) or Ghirardato, Maccheroni, Marinacci and Siniscalchi (2003) (also Chateauneuf (1994) for the special case of a linear utility function). The model has been also axiomatized in the Anscombe-Aumann framework with the outcome space being a set of (simple) lotteries over an arbitrary outcome set (see, e.g., Schmeidler, 1986, 1989, and Chateauneuf, Eichberger, Grant, 2003). As mentioned already in the introduction, this setting does not really fit the game theoretic set up. Other axiomatizations in the Savage-style setup make different structural assumption. For instance, Gilboa (1987) and Sarin and Wakker (1992) obtain only atomless capacities, which is unsuitable to games since a solution to a game may be a singleton. Wakker (1989), Nakamura (1990), and Chew and Karni (1991) apply only to a finite state space while type spaces in game theory can be large.

Instead of using some notion of support of a capacity to represent "belief" in an ad hoc way as in some prior work on ambiguity in games, we define a preference-based notion of (unambiguous) belief.

**Definition 3 (Belief)** *An event $E \in \Sigma_\Omega$ is said to be* $\succsim$-null *if, for all acts $f, g, h \in \mathcal{F}$, $f_E h \succsim g_E h$. An event $E$ is* $\succsim$-believed *if $\Omega \setminus E$ is $\succsim$-null.*

An event $E$ is not $\succsim$-*nonnull*, i.e., $f_E h \succ g_E h$ for some $f, g, h \in \mathcal{F}$.

**An Aside:** Our goal is to axiomatize CEU preferences for the measurable case.

Throughout this section, we assume that $X$ is a connected topological space and that the set of (simple) acts is endowed with a product topology (i.e., all sets $X^n$ with $n \in \mathbb{N}$).

Before we provide the axioms that justify the CEU model for the measurable case, we need additional notation. Two acts are said to be *comonotonic* (i.e., "common monotonic") if there are no two states $\omega, \omega' \in \Omega$ such that $f(\omega) \succ g(\omega)$ and $f(\omega') \prec g(\omega')$. For a partition $\pi = \{E_1, \ldots, E_n\}$ of $\Omega$, we denote by $\Sigma_\pi$ be the sub-algebra of $\Sigma_\Omega$ (i.e., the algebra of events generated by unions of events in $\pi$). $\mathcal{F}_\pi$ is the set of all acts defined on $\pi$ (i.e., all $\Sigma_\pi$-measurable acts). Hence, two acts $f$ and $g$ in $\mathcal{F}_\pi$ are comonotonic if there are no distinct numbers $i, j \in \{1, \ldots, n\}$ such that $f(E_i) \succ g(E_j)$ and $f(E_i) \prec g(E_j)$. Let $\rho(\pi)$ be a (rank-ordering) permutation on $\{1, \ldots, n\}$ induced by an act $f$ on $\pi$; i.e., $\{\rho(1) \ldots, \rho(n)\}$ such that $f(E_{\rho(1)}) \succsim \ldots \succsim f(E_{\rho(n)})$ for some $f \in \mathcal{F}_\pi$. Let $\mathcal{F}_{\rho(\pi)} = \{f \in \mathcal{F}_\pi : f(E_{\rho((1)}) \succsim \ldots \succsim f(E_{\rho(n)})\}$ be the set of comonotonic acts for a (rank-ordering) permutation $\rho(\pi)$. We call $\mathcal{F}_{\rho(\pi)}$ a $\pi$-*comoncone* (i.e., a maximal comonotonic set associated with a partition $\pi$ for a permutation $\rho(\pi)$). Notice that $\mathcal{F}_\pi$ is the union of $n!$ $\pi$-comoncons $\mathcal{F}_{\rho(\pi)}$. The set of all acts $\mathcal{F}$ is the union of all sets $\mathcal{F}_{\rho(\pi)}$.

Given a partition $\pi$, an event $E \in \pi$ is $\succsim_\pi$-*null* (i.e., w.r.t. $\succsim$ restricted to acts in $\mathcal{F}_\pi$) if for all acts $f, g, h \in \mathcal{F}_{\rho(\pi)}$, $f_E h \succsim g_E h$, otherwise $E$ is $\succsim_\pi$-*nonnull*.

The first four axioms are standard.

**Axiom 1 (Weak Order)** *$\succsim$ on $\mathcal{F}$ is a weak order (i.e., complete and transitive).*

**Axiom 2 (Monotonicty)** *For all $f, g \in \mathcal{F}$, if $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$, then $f \succsim g$.*

The next axiom requires that $\succsim$ is continuous with respect to the product topology.

**Axiom 3 (Continuity)** *For each partition $\pi = \{E_1, \ldots, E_n\}$ of $\Omega$ and act $f$ in $\mathcal{F}_\pi$, $\{(x_1, \ldots, x_n) \in X^n : f \succsim g = (E_1, x_1; \ldots; E_n, x_n)\}$ and $\{(x_1, \ldots, x_n) \in X^n : g = (E_1, x_1; \ldots; E_n, x_n) \succsim f\}$ are closed sets with respect to the product topology $X^n$.*

The next axiom, *Comonotonic-Tradeoff-Consistency*, is crucial for the CEU model. It is used to elicit equalities of utility differences, which, for cardinal utility, suffices to determine the entire utility function. To formulate the axiom, we define an indifference relation $\sim^*_{\mathcal{F}_{\rho(\pi)}}$ derived from $\succsim$. For a partition $\pi$ of $\Omega$, a $\pi$-comoncone $\mathcal{F}_{\rho(\pi)}$, and outcomes $a, b, c, d \in X$, we write

$$a \ominus b \sim^*_{\mathcal{F}_{\rho(\pi)}} c \ominus d \tag{3}$$

if there is a $\succsim_\pi$-nonnull event $E_i \in \pi$ such that

$$a_E f \sim c_E g \;\text{ and }\; b_E f \sim d_E g, \tag{4}$$

with all the four acts $a_{E_i} f, b_{E_i} g, c_{E_i} f$ and $d_{E_i} g$ being comonotonic, i.e., elements of $\mathcal{F}_{\rho(\pi)}$. The interpretation is that receiving outcome $a$ instead of $b$ is as good as as receiving $c$ instead of $d$; i.e., it exactly offsets the receipt of $f$ instead of $g$ contingent on other events in $\pi$ than $E_i$ (see Köberling and Wakker (2003)). *Comonotonic-Tradeoff-Consistency* requires that improving an outcome in any relationship $\sim^*_{\mathcal{F}_{\rho(\pi)}}$ breaks that relationship. That is, if $a \ominus b \sim^*_{\mathcal{F}_{\rho(\pi)}} c \ominus d$ then each outcome such that $a' \succ a$ precludes $a' \ominus b \sim^*_{\mathcal{F}_{\rho(\pi)}} c \ominus d$. Because of symmetry of the indifference relation $\sim^*_{\mathcal{F}_{\rho(\pi)}}$, similar conditions hold for $b, c, d$.[5]

**Axiom 4 (Comonotonic Trade-Off Consistency)** *Improving an outcome in any indifference relationship $\sim^*_{\mathcal{F}_{\rho(\pi)}}$ breaks that relationship.*

The last axiom precludes that $\succsim$ is trivial.

**Axiom 5 (Nontriviality)** *There exists at least one $\succsim$-nonnull event.*

Axioms 1 -5 are necessary and sufficient for $\succsim$ on $\mathcal{F}$ to be represented by CEU.

**Theorem 1** *$\succsim$ over $\mathcal{F}$ admits a CEU representation w.r.t. a continuous utility function $u : X \to \mathbb{R}$ and a unique capacity $\nu : \Sigma_\Omega \to \mathbb{R}$ if and only if Axioms 1-5 hold true.*

*Moreover, if there are two or more disjoint $\succsim$-nonnull events, $u$ is unique up to an affine transformation, If there are no two disjoint $\succsim$-nonnull events, $u$ is unique up to a strictly increasing transformation, $\nu$ assigns $1$ to each $\succsim$-nonnull event and $0$ otherwise.*

[5] By symmetry, we mean the following...

**Proof.** We begin the proof by deriving first few helpful lemmas.[6]

By Axiom 5, we exclude the trivial case where $f \sim g$ for all $f, g \in \mathcal{F}$. Since there is at least one $\succsim$-nonnull event, we need to distinguish two cases. In the first, non-degenerate case, there are at least two disjoint $\succsim$-nonnull events. This means that there exists a partition $\pi$ that contains at least two $\succsim_\pi$-nonnull events. In the second, degenerate case, there are no two disjoint $\succsim$-nonnull events. This means that each partition $\pi$ contains exactly one $\succsim_\pi$-believed event. For this case, Lemma 3 shows $\succsim$ over $\mathcal{F}_\pi$ admits a CEU representation with respect to a (unique) capacity $\nu_\pi$ on $\Sigma_\pi$ and an (ordinal) utility function $u_\pi$ (i.e., a utility function that is unique up to an increasing transformation).

Lemma 1 and 2 apply to the non-degenerate case. Lemma 1 shows that for each partition $\pi$ containing at least two $\succsim$-nonnull events, the preferences $\succsim$ over $\mathcal{F}$ restricted to $\mathcal{F}_\pi$, denoted by $\succsim_\pi$, admits a CEU representation with respect to a (unique) capacity $\nu_\pi$ on $\Sigma_\pi$ and a (cardinal) utility function $u_\pi$ (i.e., a utility function that is unique up to an affine transformation). Lemma 2 shows that for two partitions $\pi'$ and $\pi''$, the respective CEU representations "fit together", i.e., $\succsim$ over $\mathcal{F}_{\pi'}$ and $\succsim$ over $\mathcal{F}_{\pi''}$ are represented with respect to the "same" utility function $u$ and furthermore, $\nu_{\pi'}$ and $\nu_{\pi''}$ agree on the same events; that is, for each $E_i \in \pi$ and $F_j$ in $\pi'$ such that $E_i = F_j$, $\nu_\pi(E_i) = \nu_{\pi'}(F_j)$. This implies that the CEU representation of $\succsim$ over $\Sigma_\pi$-measurable acts is independent of $\pi$.

**Lemma 1** *Let $\pi = \{E_1, \ldots, E_n\}$ be a partition and $\mathcal{F}_\pi$ be the set of $\Sigma_\pi$-measurable acts. Suppose that $\pi$ contains at least two $\succsim$-nonnull events and that $\succsim$ over $\mathcal{F}$ satisfies Axioms 1 through 4. Then, $\succsim$ over $\mathcal{F}_\pi$ admits a CEU representation; i.e., there exist a capacity $\nu_\pi : \Sigma_\pi \to \mathbb{R}$ and a continuous utility function $u_\pi : X \to \mathbb{R}$ such that for all $f, g \in \mathcal{F}_\pi$:*

$$f \succsim g \quad \text{if and only if} \quad \int_\Omega u_\pi(f) d\nu_\pi \geq \int_\Omega u_\pi(g) d\nu_\pi. \tag{5}$$

*Moreover, $u_\pi$ is unique up to an affine transformation and $\nu_\pi$ is uniquely determined.*

**Proof.** Notice that for each partition $\pi = \{E_1, \ldots, E_n\}$, the set $\mathcal{F}_\pi$ of $\Sigma_\pi$-measurable acts is isomorphic to the set of acts defined over a (finite) set of states with $n$ elements. Thus, we can apply Corollary 8 of Theorem 8 in Köbberling and Wakker (2003) showing that, under Axioms 1-3 and (modified) 4, there exists a unique capacity $\nu_\pi : \Sigma_\pi \to \mathbb{R}$ and a continuous (cardinal) utility function $u_\pi : X \to \mathbb{R}$ that represent $\succsim$ over $\mathcal{F}_\pi$ via (5). ■

**Lemma 2** *Let $\pi'$ and $\pi''$ be two partitions of $\Omega$, each containing at least two $\succsim$-nonnull events. Let $(\nu_{\pi'}, u_{\pi'})$ and $(\nu_{\pi''}, u_{\pi''})$ be the respective CEU representations of $\succsim$ over $\mathcal{F}_{\pi'}$ and $\succsim$ over $\mathcal{F}_{\pi''}$ (as obtained in Lemma 1). Then, $u_{\pi'}$ is an affine transformation of $u_{\pi''}$. Moreover, $\nu_\pi(E) = \nu_{\pi'}(F)$ if $E = F$ for some $E \in \Sigma_\pi$ and $F \in \Sigma_{\pi'}$.*

[6] We denote by $\bar{\pi}$, a fixed partition and by $\mathcal{F}_{\rho(\bar{\pi})}$, the set of comonotonic acts for a permutation $\rho(\bar{\pi})$.

**Proof.** Consider two partitions $\pi' = \{E_1, \ldots, E_m\}$ and $\pi'' = \{F_1, \ldots, F_n\}$. Let $\pi$ be another partition defined as the join (i.e., the coarsest common refinement) of $\pi'$ and $\pi''$:

$$\pi = \{E_1 \cap F_1, \ldots, E_1 \cap F_n, \ldots, E_m \cap F_1, \ldots, E_m \cap F_n\} \tag{6}$$

Suppose, without loss of generality, that $E_1$ and $E_2$ are $\succsim$-nonnull events. Then, at least one $E_1 \cap F_j$ with $j = 1, \ldots, n$ is $\succsim$-nonnull. Likewise, at least $E_2 \cap F_j$ with $j = 1, \ldots, n$ is $\succsim$-essential. Hence, $\pi$ contains at least two $\succsim$-nonnull events. Thus, by Lemma 1, the preference relation $\succsim$ over $\mathcal{F}_\pi$ admits a CEU representation with respect to a unique $\nu_\pi$ and a continuous utility function $u_\pi$ that is unique up to an affine transformation.

Consider constant acts (i.e., acts $f \in \mathcal{F}_\pi$ such that $f^{-}(x) = \Omega$ for all $x \in X$). For each $x, y \in X$, $x \succsim y$ or $x \precsim y$, by completeness. Suppose $x \succsim y$. By the previous step,

$$x \succsim y \text{ iff } \int_\Omega u_\pi(x) d\nu_\pi = u_\pi(x)\nu_\pi(G) \;\; \geq \;\; u_\pi(y)\nu(G) = \int_\Omega u_\pi(y) d\nu_\pi \tag{7}$$

$$u_\pi(x) \;\; \geq \;\; u_\pi(y) \tag{8}$$

where $G \in \Sigma_\pi$ is a $\succsim_\pi$-believed event. Hence, constant acts are ranked by $u_\pi$. From the CEU representation $(\nu_{\pi'}, u_{\pi'})$ of $\succsim$ over $\mathcal{F}_{\pi'}$, we have $x \succsim y$ if and only if $u_{\pi'}(x) \geq u_{\pi'}(y)$. Since $\mathcal{F}_\pi$ and $\mathcal{F}_{\pi'}$ contain all constant acts, for all $x, y \in X$, $u_\pi(x) \geq u_\pi(y)$ if an only if $u_{\pi'}(x) \geq u_{\pi'}(y)$ if and only if $u_{\pi'} = \alpha + \beta u_\pi$ for each $a \geq 0$ and $b > 0$. Hence, $u_{\pi'}$ is an affine transformation of $u_\pi$. The same argument applies to $u_{\pi''}$, showing that $u_{\pi''}$ is an affine transformation of $u_\pi$. Thus, all utility functions $u_\pi$, $u_{\pi'}$, and $u_{\pi''}$ are affine transformation of each other. Hence, we may choose one representation, e.g., let $u := u_\pi$.

Suppose $A \in \Sigma_{\pi'}$ and $B \in \Sigma_{\pi''}$ are such that $A = B$. Hence, $A = \bigcup_{i=1}^m E_i \cap A$ and $B = \bigcup_{j=1}^n F_j \cap B$. Clearly, $A$ and $B$ are in $\Sigma_\pi$ since $A = \bigcup_{j=1}^n A \cap F_j$ and $B = \bigcup_{i=1}^m B \cap E_i$. Since $\nu_\pi$ on $\Sigma_\pi$ is uniquely determined, $\nu_\pi(A) = \nu_\pi(B)$.

Each $\Sigma_{\pi'}$-measurable act is $\Sigma_\pi$-measurable (i.e., $\mathcal{F}_{\pi'} \subset \mathcal{F}_\pi$). Since such acts are ranked by the same preference relation $\succsim$, for each act $f \in \mathcal{F}_{\pi'} \cap \mathcal{F}_\pi$, we have

$$\int_\Omega u(f) d\nu_\pi = \int_\Omega u(f) d\nu_{\pi'}, \tag{9}$$

implying that $\nu_\pi(E) = \nu_{\pi'}(E)$ for all $E \in \Sigma_{\pi'} \cap \Sigma_\pi$. Similar, since $\mathcal{F}_{\pi''} \subset \mathcal{F}_\pi$, $\nu_\pi(F) = \nu_{\pi''}(F)$ for all $F \in \Sigma_{\pi''} \cap \Sigma_\pi$. Hence, since $\nu_{\pi'}$ and $\nu_{\pi''}$ are unique, $\nu_\pi(A) = \nu_\pi(B)$, $\nu_\pi(A) = \nu_{\pi'}(A)$ and $\nu_\pi(B) = \nu_{\pi''}(B)$, we thus get $\nu_{\pi'}(A) = \nu_{\pi''}(B)$, as desired. ∎

**Lemma 3** *Let $\pi = \{E_1, \ldots, E_n\}$ be a partition and $\mathcal{F}_\pi$ be the set of $\Sigma_\pi$-measurable acts. Suppose that $\pi$ does not contain two $\succsim$-nonnull events. Under Axioms* 1-5*, $\succsim$ over $\mathcal{F}_\pi$ admits a CEU representation with respect to a continuous utility function $u_\pi : X \to \mathbb{R}$ and a unique capacity $\nu_\pi : \Sigma_\pi \to \mathbb{R}$ assigning $\nu(E) = 1$ for each $\succsim$-nonnull event $E$, and $\nu(E) = 0$ otherwise. Moreover, $u_\pi$ is unique up a strictly increasing transformation.*

**Proof.** When each $\succsim$-nonnull event is $\succsim$-believed, the existence of a utility function on the outcome space follows from Theorem 2.2 of Krantz, Luce, Suppes and Tversky (1971), ■

We summarize our lemmas and prove Theorem 1. The case where $\Sigma_\Omega$ does not contain two disjoint $\succsim$-nonnull events is treated in Lemma 3. So, let us assume that $\Sigma_\Omega$ contains at least two disjoint $\succsim$-nonnull events. By Lemma 1, for each partition $\pi = \{E_1, \ldots, E_n\}$ of $\Omega$ that contains at least two disjoint $\succsim$-nonnull events, there exist a (unique) capacity $\nu_\pi$ on $\Sigma_\pi$ and a continuous utility function $u_\pi : X \to \mathbb{R}$ that represent the preference relation $\succsim$ over all $\Sigma_\pi$-measurable acts in $\mathcal{F}$ (i.e., acts of the form $f = (E_1, x_1; \ldots; E_n, x_n$ ) via CEU. By Lemma 2, $\nu_\pi$ and $u_\pi$ are independent of $\pi$. Hence, we can construct $\nu$ on $\Sigma_\Omega$ as follows: For each $A \in \Sigma_\Omega$, $\nu(A) := \nu_\pi(A)$ for a partition $\pi$ of $\Omega$ such that $A \in \Sigma_\pi$.

Obviously, $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$. To show that $\nu$ is monotone, take two events $A, B \in \Sigma_\Omega$ such that $A \subseteq B$. Let $\pi'$ and $\pi''$ be two partitions such that $A = E_i$ and $B = E_j$ for some $E_i \in \pi$ and $F_j \in \pi'$. Let one of the partitions contain at least two disjoint $\succsim$-nonnull events. Let $\pi$ be the joint of $\pi'$ and $\pi''$. By a similar arguments as in Lemma 2, $\pi$ contains at least two disjoint $\succsim$-nonnull events. Let $\nu_\pi$ be the capacity on $\Sigma_\pi$ implied by Lemma 1. Since $\nu_\pi$ is monotone on $\Sigma_\Omega$, we have $\nu_\pi(\bigcup_{j=1}^n E_i \cap F_j) \leq \nu_\pi(\bigcup_{i=1}^n E_i \cap F_j)$, or equivalently, $\nu_\pi(A) \leq \nu_\pi(B)$. Thus, $\nu(A) \leq \nu(B)$, showing that $\nu$ is monotone on $\Sigma_\Omega$.

Hence, for any two acts $f, g \in \mathcal{F}$, ■

Now, we summarize the construction. we have $\nu_\pi(\bigcup_{j=1}^n E_i \cap F_j) = \nu_{\pi'}(E_i)$ and thus $\nu_\pi(\bigcup_{i=1}^m E_i \cap F_j) = \nu_{\pi''}(F_j)$.

Hence, $\nu_{\tilde{\pi}}(E) = \nu_\pi(E)$ and $\nu_{\tilde{\pi}}(F) = \nu_{\pi'}(F)$ and thus $\nu_\pi(E) = \nu_{\pi'}(F)$.

`Adam, fill in details on K¨bberling and Wakker for measurable spaces?`

While a preference-based notion of belief is conceptually important when "importing" decision theory into game theory, it is useful in applications to have a characterization of belief at the level of capacities.[7]

## 2.2 Unambiguously Believed Events

**Proposition 1** *Let $\succeq$ be a Choquet expected utility preference with respect to a capacity $\nu$ on $\Sigma_\Omega$. The following statements are equivalent:*

*(i)* *Event $E \in \Sigma_\Omega$ is $\succsim$-believed.*

*(ii)* $\nu((\Omega \setminus E) \cup F) = \nu(F)$ *for all events* $F \in \Sigma_\Omega$, $F \subseteq E$.

*(iii)* $\nu(G \cup F) = \nu(F)$ *for all* $F, G \in \Sigma_\Omega$ *with* $G \subseteq \Omega \setminus E$.

From now on, we take statement *(ii)* quasi as a definition of belief.

[7]This result comes from an earlier unfinished project of Amanda Friedenberg and Burkhard.

While our notion of belief goes back essentially to subjective expected utility à la Savage (1954), it has also been used by Epstein (1997), Morris, (1997), Ghiradato and Le Breton (1999), and Chen and Luo (2012) when considering preferences allowing for ambiguity in an interactive setting.

The notion of belief is closely related to the preference-based notion of "unambiguous events". More precisely, if an event is believed, it is an unambiguous event in the sense of Sarin and Wakker (1992) and Nehring (1999). An event $E \in \Sigma_\Omega$ is unambiguous if $\succsim$ satisfies the Sure-Thing Principle constrained to $E$ and $\Omega \setminus E$. The following definition is due to Sarin and Wakker (1992).

**Definition 4 (Unambiguous Event)** *An event $E \in \Sigma_\Omega$ is said to be $\succsim$-unambiguous if, for any $f, g, h, h' \in \mathcal{F}$,*

$$\begin{array}{rcll} f_E h \succsim g_E h & \text{if and only if} & f_E h' \succsim g_E h', & \text{and} \\ f_{\Omega\setminus E} h \succsim g_{\Omega\setminus E} h & \text{if and only if} & f_{\Omega\setminus E} h' \succsim g_{\Omega\setminus E} h'. & \end{array}$$

*Otherwise, $E$ is called $\succsim$-ambiguous.*

In terms of a capacity $\nu$, unambiguous events admit the following characterization. This characterization has been proved previously in a finite set up by Dominiak and Lefort (2011).[8] Our proof in the appendix extends Dominiak and Lefort (2011, Proposition 3.1) to the measurable case.

**Proposition 2** *Let $\succeq$ be a Choquet expected utility preference with respect to a capacity $\nu$ on $\Sigma_\Omega$. An event $E \in \Sigma_\Omega$ is $\succsim$-unambiguous if and only if, for any $F \in \Sigma_\Omega$,*

$$\nu(F) = \nu(F \cap E) + \nu(F \cap (\Omega \setminus E)). \tag{10}$$

Condition (10) says that the capacity $\nu$ is *additively-separable* across the unambiguous events. Intuitively, one would expect that there is a close relationship between additivity of a capacity and unambiguous events. However, as pointed out by Nehring (1999), we know that the standard additivity condition is not sufficient for an event to be unambiguous (unless the capacity is convex). Example 6 in the appendix illustrates, additivity of a capacity w.r.t. to an event does not imply that the event is perceived unambiguous.

Also, additivity of a capacity on every (bi)partition $\{E, \Omega \setminus E\}$, i.e., $\nu(E)+\nu(\Omega \setminus E) = 1$ for any $E \in \mathcal{F}_E$ does not imply that *all* events are unambiguous. This in turn means that if there exists an ambiguous event w.r.t. to a capacity $\nu$, it does not imply that there exists an event $E \in \mathcal{F}_E$ on which the capacity is non-additive, i.e., $\nu(E) + \nu(\Omega \setminus E) \neq 1$. This is illustrated in Example 7 in the appendix.

We have that an event is believed if and only if it is an unambiguous event with the capacity value 1.

[8] Nehring (1999) provides another characterization of unambiguous events in a finite setup with a linear utility function.

**Proposition 3** *Let $\succeq$ be a Choquet expected utility preference with a capacity $\nu$ on $\Sigma_\Omega$. The following statements are equivalent:*

*(i) Event $E \in \Sigma_\Omega$ is $\succsim$-believed.*

*(ii) Event $E$ is $\succsim$-unambiguous with $\nu(E) = 1$.*

*(iii) Event $\Omega \setminus E$ is $\succsim$-unambiguous with $\nu(\Omega \setminus E) = 0$.*

In Appendix A.8 we relate our notion of belief to notions of support of a capacity that have been studied previously in the game-theoretic literature. It is clear that unlike for Subjective Expected Utility theory, these notions of support for capacities are not necessarily equivalent to the notion of belief.

## 2.3 Properties of Belief

Our epistemic characterization is facilitated by standard properties of beliefs.

**Proposition 4** [9] *Let $\succeq$ be a Choquet expected utility preference w.r.t. to a capacity $\nu$ on $\Sigma_\Omega$. Then*

*Necessitation: $\Omega$ is $\succeq$-believed.*

*Monotonicity: If $E \subseteq F$, $E, F \in \Sigma_\Omega$, then $E$ being $\succeq$-believed implies that $F$ is $\succeq$-believed.*

*Conjunction I: Let $E_1, E_2, \ldots \in \Sigma_\Omega$. If $\bigcap_j E_j$ is $\succeq$-believed, then $E_1, E_2, ..$ are each $\succeq$-believed.*

*Finite Conjunction: Let $E_1, \ldots, E_n \in \Sigma_\Omega$ be $\succeq$-believed. Then $\bigcap_{i=1}^n E_i$ is $\succeq$-believed.*

Note that we just have finite conjunction. It can be the case that measurable events $E_1, E_2, \ldots$ may be $\succeq$-believed, even though $\bigcap_i E_i$ is not $\succeq$-believed. See the appendix for an example.[10] However, the following continuity assumption on capacities yields the "full" conjunction property of belief. A capacity $\nu$ is *lower continuous* if

$$\lim_{n\to\infty} \nu(E_n) = \nu\left(\bigcup_{n=1}^{\infty} E_n\right),$$

for any sequence of events $E_1, E_2, ... \in \Sigma_\Omega$ with $E_n \subseteq E_{n+1}$. While this assumption could be stated in terms of the primitive preference $\succeq$, it is essentially impossible to test it behaviorally. That's why we impose it directly on the capacity and not on the underlying preference relation $\succeq$.

[9] These results stem from an earlier unfinished project of Amanda Friedenberg and Burkhard.
[10] This example due to Amanda Friedenberg.

**Lemma 4** [11] *Let $\succeq$ be a Choquet expected utility preference w.r.t. to a lower continuous capacity $\nu$ on $\Sigma_\Omega$. Further, consider any sequence of events $E_n \in \Sigma_\Omega$, $n = 1, 2, \ldots$, with $E_{n+1} \subseteq E_n$. If each $E_1, E_2, \ldots$ is $\succeq$-believed, then $\bigcap_{n=1}^{\infty} E_n$ is $\succeq$-believed.*

In the appendix we show that replacing lower continuity with regularity may let conjunction fail.[12]

# 3 Choquet Rationalizability

Fix a finite strategic game form $\langle I, (A_i)_{i \in I}, (o_i)_{i \in I} \rangle$, for which $I$ is a nonempty finite set of players and for each player $i \in I$, $A_i$ is a nonempty finite set of actions and $o_i : A \longrightarrow \mathbb{X}$ is the outcome function with $A := \times_{i \in I} A_i$ that assigns to each profile of actions $\mathbf{a} \in A$ an outcome $o_i(\mathbf{a})$ in the previously introduced outcome space $X$. As usual, for any collection of sets $(Y_i)_{i \in I}$ we denote by $Y = \times_{i \in I} Y_i$ and $Y_{-i} = \times_{j \in I \setminus \{i\}} Y_j$ with generic elements $\mathbf{y}$ and $y_{-i}$, respectively.

Next we connect Choquet expected utility theory to games in strategic form. Given a strategic game form $\langle I, (A_i), (o_i) \rangle$, for any player $i \in I$ and any action $a_i \in A_i$ we denote by $f^{a_i} : A_{-i} \longrightarrow X$ the act of player $i$ associated with action $a_i$ defined by $f^{a_i}(a_{-i}) := o_i(a_i, a_{-i})$. The set of opponents' action profiles $A_{-i}$ takes on the role of the state space in Choquet expected utility theory signifying the fact we model strategic uncertainty. The strategic game form $\langle I, (A_i), (o_i) \rangle$ together with the utility functions $(u_i)_{i \in I}$ over outcomes in $X$ define a game in strategic form $\langle I, (A_i), (u_i \circ o_i) \rangle$.

Let $\nu_i$ be a capacity on $A_{-i}$. We say that $a_i^* \in A_i$ is a *Choquet best response* to $\nu_i$ if

$$a_i^* \in \arg\max_{a_i \in A_i} \int_{A_{-i}} u_i(o_i(a_i, a_{-i})) d\nu_i(a_{-i}), \tag{11}$$

where the integral is the Choquet integral defined above.

Denote by $\mathcal{C}(A_{-i})$ the set of all capacities on $A_{-i}$.

**Definition 5 (Choquet rationalizability)** *For $i \in I$ and $k \geq 1$ define inductively,*

$$\begin{aligned}
C_i^1 &= \mathcal{C}(A_{-i}) \\
R_i^1 &= \left\{ a_i \in A_i : \begin{array}{l} \text{there exists } \nu_i \in C_i^1 \text{ for which } a_i \\ \text{is a Choquet best response} \end{array} \right\} \\
&\vdots \\
C_i^{k+1} &= \left\{ \nu_i \in C_i^k : \begin{array}{l} \nu_i((A_{-i} \setminus R_{-i}^k) \cup F) = \nu_i(F) \\ \text{for all } F \subseteq R_{-i}^k \end{array} \right\} \\
R_i^{k+1} &= \left\{ a_i \in A_i : \begin{array}{l} \text{there exists } \nu_i \in C_i^{k+1} \text{ for which} \\ a_i \text{ is a Choquet best response} \end{array} \right\}
\end{aligned}$$

[11] This lemma stem from an earlier unfinished project of Amanda Friedenberg and Burkhard.

[12] The example and the arguments are due to Amanda Friedenberg.

*The set of Choquet rationalizable actions is*

$$R_i^\infty = \bigcap_{k=1}^{\infty} R_i^k.$$

Choquet rationalizability is defined as a reduction procedure on sets of capacities. It implies a reduction procedure on sets of actions for each player.

Alternatively we can consider a "fixed-point" definition suggested verbally in the last section of Ghirardato and Le Breton (1999, p. 15).

**Definition 6 (Fixed-point definition)** *Define $(R_i)_{i \in I}$ with $R_i \subseteq A_i$ for $i \in I$ to be the largest set such that every $a_i \in R_i$ is a Choquet best response with respect to a capacity $\nu_i \in \mathcal{C}(A_{-i})$ satisfying $\nu_i((A_{-i} \setminus R_{-i}) \cup F) = \nu_i(F)$ for all $F \subseteq R_{-i}$.*

**Remark 1** *If $(R_i)_{i \in I}$ and $(\tilde{R}_i)_{i \in I}$ are two collections of sets, each satisfying Definition 6, then $R_i = \tilde{R}_i$ for all $i \in I$.*

In light of Remark 1 we are justified to refer to any $(R_i)_{i \in I}$ as *the* largest set satisfying the property of Definition 6.

The fixed-point definition of Choquet rationalizability is equivalent to the inductive definition.

**Theorem 2** *For any finite game in strategic form, $R_i = R_i^\infty$ for all $i \in I$,.*

This result parallels the equivalence of the fixed-point definition and inductive definition of standard rationalizability à la Bernheim (1984) and Pearce (1984).

## 3.1 Iterative Dominance in Extended Games

Choquet rationalizability is a reduction procedure on beliefs represented by capacities. In applications it is sometimes easier to use a reduction procedure on actions instead. Moreover, the computation of Choquet expected utilities may be viewed as an impediment to applications of Choquet expected utility theory in games. Fortunately, we can characterize Choquet rationalizability by an iterated elimination procedure akin to iterated elimination of strictly dominated actions a suitably extended game that does not require the computation of the Choquet integral. In this extended game, a player's set of actions is the set of nonempty subsets of actions in the underlying. Moreover, payoffs for any combinations of subsets of actions in the underlying game correspond to the minimum payoffs achieved with actions in the subset. The precise definition is as follows:

**Definition 7 (Extended Game)** *Given a game in strategic form $G = \langle I, (A_i)_{i \in I}, (u_i \circ o_i)_{i \in I} \rangle$, we define an associated extended game $\mathcal{G} = \langle I, (\mathcal{A}_i)_{i \in I}, (\tilde{u}_i)_{i \in I} \rangle$ in which the set of players is the set of players $I$ in the underlying game $G$, player $i$'s set of actions $\mathcal{A}_i := 2^{A_i} \setminus \{\emptyset\}$ is the set of nonempty subsets of actions of the underlying game $G$, and player $i$'s utility function $\tilde{u}_i : \mathcal{A} \longrightarrow \mathbb{R}$ is defined by $\tilde{u}_i(A'_i, A'_{-i}) = \min_{\mathbf{a} \in A'_i \times A'_{-i}} u_i(o_i(\mathbf{a}))$ for all $(A'_i, A'_{-i}) \in \mathcal{A} := \times_{i \in I} \mathcal{A}_i$.*

The following example illustrates the construction of extended games.

**Example 1** *Consider the following $3 \times 2$ game (left) and the associated extended game (right):*

| $G$ | $l$ | $r$ |
|---|---|---|
| $u$ | $4,0$ | $0,4$ |
| $d$ | $0,4$ | $4,0$ |
| $m$ | $1,2$ | $1,2$ |

| $\mathcal{G}$ | $\{l\}$ | $\{r\}$ | $\{l,r\}$ |
|---|---|---|---|
| $\{u\}$ | $4,0$ | $0,4$ | $0,0$ |
| $\{d\}$ | $0,4$ | $4,0$ | $0,0$ |
| $\{m\}$ | $1,2$ | $1,2$ | $1,2$ |
| $\{u,d\}$ | $0,0$ | $0,0$ | $0,0$ |
| $\{u,m\}$ | $1,0$ | $0,2$ | $0,0$ |
| $\{d,m\}$ | $0,2$ | $1,0$ | $0,0$ |
| $\{u,d,m\}$ | $0,0$ | $0,0$ | $0,0$ |

We like to stress that we view the extended game as a technical device that facilitates computing Choquet rationalizable actions without the need to compute Choquet expected utility in games. Although we do not champion this interpretation, one may interpret the extended game as a game in which players can chose ambiguous actions in the sense of choosing non-singleton subsets of actions.

A subset $Y \subseteq A$ is called a *restriction of player $i$* (or an $i$-product set) if $Y = Y_i \times Y_{-i}$ for some $Y_i \subseteq A_i$ and $Y_{-i} \subseteq A_{-i}$. Clearly, $A$ itself is a restriction for every player $i \in N$. Given a restriction $Y = Y_i \times Y_{-i}$ of player $i$ in the game $G$, the associated restriction in the associated extended game $\mathcal{G}$ is defined by $\mathcal{Y} = \mathcal{Y}_i \times \mathcal{Y}_{-i}$ where $\mathcal{Y}_i = 2^{Y_i} \setminus \{\emptyset\}$ and $\mathcal{Y}_{-i} = 2^{Y_{-i}} \setminus \{\emptyset\}$.

For each player $i \in I$, let $\mathcal{A}_i^\circ \subseteq \mathcal{A}_i$ denote the subset of singleton subsets in $\mathcal{A}_i$. These are the actions in the extended game that actually correspond to actions in the underlying game.

**Definition 8 (Strict Domination in Extended Games)** *A subset of actions $A'_i \in \mathcal{A}_i$ is* strictly dominated in player *$i$'s* restriction $\mathcal{Y} \subseteq \mathcal{A}$ by a mixed action *in the extended game $\mathcal{G}$ if $A'_i \in \mathcal{Y}_i$, $\mathcal{Y}_{-i} \neq \emptyset$ and there exists $a_i \in A'_i$ for which there exists a mixed action $\alpha_i \in \Delta(\mathcal{Y}_i \cap \mathcal{A}_i^\circ)$ such that*

$$\tilde{u}_i(\alpha_i, A'_{-i}) > \tilde{u}_i(\{a_i\}, A'_{-i}) \text{ for all } A'_{-i} \in \mathcal{Y}_{-i},$$

*where $\Delta(\mathcal{Y}_i \cap \mathcal{A}_i^\circ)$ denotes the set of probability measures on $\mathcal{Y}_i \cap \mathcal{A}_i^\circ$ and (with some abuse of notation) $\tilde{u}_i(\alpha_i, A'_{-i})$ is player $i$'s expected utility from playing the mixed action $\alpha_i$ when $i$'s opponents play $A'_{-i}$ in the extended game.*

Three remarks are in order: First, a nonempty subset of actions is strictly dominated whenever there is an action (an object of choice that can actually be played by the player) in this set that is strictly dominated. Second, any mixed action that dominates an action mixes only over singleton action sets in the extended game. Thus, they do correspond to mixtures in the underlying game. Third, the importance of the extended game comes from how payoffs are evaluated. In particular, the mixed action dominating an action must yield a *strictly* larger payoff also for non-singleton action profiles of opponents. This suggests that there might be actions that are strictly dominated in the underlying game (with respect to a mixed action) but are not eliminated in the extended game. We will explore this aspect in Section 4 when we study the effect of ambiguity attitudes.

**Definition 9 (Never Choquet Best Response)** *We say that an action $a_i$ is* never a Choquet best response *on player* $i$'s *restriction $Y$ if there does not exist a capacity $\nu_i \in \mathcal{C}(Y_{-i})$ for which it is a Choquet best response.*

The following lemma is the analogue to Pearce (1984, Lemma 3) for Choquet expected utility.

**Lemma 5** *Given a finite game in strategic form $G = \langle I, (A_i)_{i \in I}, (u_i \circ o_i)_{i \in I} \rangle$, action $a_i \in A_i$ is never a Choquet best response on player i's restriction $Y$ if and only if $\{a_i\}$ is strictly dominated in player i's associated restriction $\mathcal{Y}$ of the associated extended game $\mathcal{G} = \langle I, (\mathcal{A}_i)_{i \in I}, (\tilde{u}_i)_{i \in I} \rangle$.*

The proof follows directly from Ghirardato and Le Breton (1999, Theorems 1 and 2). They consider single person decision problems under ambiguity. Their notion of an act being Choquet rational corresponds in our setting to there exists a capacity with which the corresponding action is a Choquet best response. Their notion of extended decision problem corresponds to our notion of extended game in the single player case. In their Theorem 2 they show that an act is Choquet rational (w.r.t to any capacity) if and only if it is Shafer rational.[13] They also show in their Theorem 1 that an act is Shafer rational if and only if is not strictly dominated by mixtures in the extended problem. Their notion of strict domination corresponds to our notion of strict domination in the extended game in the case of single players and singleton action sets.

**Definition 10 (Iterated Strict Dominance)** *For every player $i \in I$ and every of player i's extended restriction $\mathcal{Y} \subseteq \mathcal{A}$ define*

$$U_i(\mathcal{Y}) := \{A'_i \in \mathcal{A}_i \mid A'_i \textit{ is not strictly dominated in } Y\}.$$

*Define now inductively for $i \in I$ and $k \geq 0$,*

[13] An act is Shafer rational if it is rational w.r.t. a special capacity called a belief function.

$U_i^0(\mathcal{A}) = \mathcal{A}_i$
$U_i^{k+1}(\mathcal{A}) = U_i(U^k(\mathcal{A}))$ *for* $k \geq 0$
$U_i^\infty(\mathcal{A}) = \bigcap_{k=0}^\infty U_i^k(\mathcal{A})$.

$U^\infty(\mathcal{A})$ *is called the maximal reduction. It is the set of profiles of action sets that survive iterated elimination of strictly dominated action sets (IESDA) in the extended game.*

For any extended restriction $\mathcal{Y}$, $U(\mathcal{Y}) = \times_{i \in I} U_i(\mathcal{Y})$ is an extended restriction of every player. Note that when we defined the operator $U_i$ on player $i$'s extended restrictions, we allowed $A'_i \in \mathcal{A}_i$ (instead of requiring that $A'_i$ is in player $i$'s extended restriction). The following property holds:

**Remark 2** *For any player* $i \in I$ *and* $k \geq 0$, $U_i^{k+1}(\mathcal{A}) \subseteq U_i^k(\mathcal{A})$.

We show that Choquet rationalizability is characterized by iterated eliminated of strictly dominated actions in the associated extended game.

For $i \in I$ and $k \geq 0$, define $A_i^k = \{a_i \in A_i \mid a_i \in A'_i \text{ for some } A'_i \in U_i^k(\mathcal{A})\}$ and $A_i^\infty = \{a_i \in A_i \mid a_i \in A'_i \text{ for some } A'_i \in U_i^\infty(\mathcal{A})\}$. $A_i^k$ are the actions of player $i$ that survive $k$-levels of iterate elimination of strictly dominated actions in the associated extended game. The following remark follows immediately from the definition of strict dominance in the extended game.

**Remark 3** *For any* $i \in I$, $a_i \in A_i^k$ *if and only if* $\{a_i\} \in U_i^k(\mathcal{A})$ *for any* $k \geq 0$ *and* $a_i \in A_i^\infty$ *if and only if* $\{a_i\} \in U_i^\infty(\mathcal{A})$.

We are ready to state our characterization result: level-$k$ Choquet rationalizable actions are characterizes by $k$-level iterative elimination of strictly dominated actions in the extended game. Moreover, Choquet rationalizable actions are equivalent to iterative elimination of strictly dominated actions in the extended game.

**Theorem 3** *For any finite strategic game, any player* $i \in I$, *and* $k \geq 1$, $R_i^k = A_i^k$ *and* $R_i^\infty = A_i^\infty$.

## 3.2 Representation-based Rationalizability Notions

This section focuses on a special case of a seminal paper by Epstein (1997). He introduced a representation-based rationalizability concept for games with general preferences. Although his class of preferences include Choquet expected utility, the case of Choquet expected utility has not been developed rigorously. We fill in the details.

To defined Epstein's (1997) representation-based notion for the case of Choquet rationalizability, let $\mathcal{R}^{u_i}(A_{-i})$ be the set of Choquet expected utility functions evaluating acts defined on $A_{-i}$ given the utility function $u_i$ on outcomes in $X$. Moreover, we write $\mathcal{R}^{u_i}(A_{-i} \mid E)$ for player $i$'s set of Choquet expected utility functions that believe the event $E \subseteq A_{-i}$. More precisely, $\mathcal{R}^{u_i}(A_{-i} \mid E)$ is the set of Choquet expected utility functions that correspond to a preference $\succeq$ for which the event $E$ is $\succeq$-believed.[14]

The following definitions specialize Epstein's (1997) rationalizability notions to the case of Choquet expected utility:

**Definition 11 (Representation-based)** *For $i \in I$, define inductively,*

$$E_i^0 = A_i$$

*and for $k \geq 1$,*

$$E_i^k = \left\{ a_i \in A_i : \begin{array}{l} \text{There exist } CEU_i \in \mathcal{R}^{u_i}(A_{-i} \mid E_{-i}^{k-1}) \text{ s.t.} \\ CEU_i(f^{a_i}) \geq CEU_i(g) \text{ for any } g \in \mathcal{F}^{A_i} \end{array} \right\}$$

*The set of representation-based Choquet rationalizable actions is defined by*

$$E_i^\infty = \bigcap_{k=0}^{\infty} E_i^k.$$

Epstein (1997) also provided an alternative "fixed"-point definition, which we phrase for the case of Choquet expected utility as follows:

**Definition 12 (Fixed-point definition)** *Define $(E_i)$ with $E_i \subseteq A_i$ for $i \in I$ to be the largest set such that for every $a_i \in E_i$ there exist a Choquet expected utility function $CEU_i \in \mathcal{R}^{u_i}(A_{-i} \mid E_{-i})$ such that $CEU_i(f^{a_i}) \geq CEU_i(g)$ for all $g \in \mathcal{F}^{A_{-i}}$.*

Epstein (1997, Theorem 3.2) implies the equivalence of both notions:

**Theorem 4 (Epstein, 1997)** *For any finite strategic game and any player $i \in I$, $E_i = E_i^\infty$ for all $i \in I$.*

We verify that Epstein's notion applied to the case of Choquet expected utility is indeed equivalent to our notion Choquet rationalizability.

**Theorem 5** *For any finite game in strategic form, any player $i \in I$, and $k \geq 1$, $R_i^k = E_i^k$ and $R_i^\infty = E_i^\infty$.*

Epstein (1997, Theorem 3.2) proved nonemptyness of $E_i$ for all $i \in I$. Together with Theorem 5 it implies Theorem **??** as a corollary.

[14] Epstein (1997) uses the term knowledge instead of belief but he defines it as in our setting as complement to Savage null.

# 4 Common Belief in Choquet Rationality

In this section, we provide an epistemic characterization of Choquet rationalizability. To this end, we introduce type spaces that allow us to formalize each player's belief over other player's behavior, their beliefs, etc.

Fix a game in strategic form $\langle I, (A_i), (u_i \circ o_i)\rangle$. A *capacity-type space* is a tuple $\langle (T_i)_{i\in I}, (s_i)_{i\in I}, (\tau_i)_{i\in I}\rangle$ with $T_i$ being player $i$'s measurable space of types, $s_i : T_i \longrightarrow A_i$ a measurable strategy mapping, and $\tau_i : T_i \longrightarrow \mathcal{C}(T_{-i})$ being player $i$'s measurable type mapping that maps each type to a capacity over opponents' types. The strategy mapping $s_i$ should not be interpreted as an object of choice of player $i$. Rather, it is just a device that allows us to specify for each type which action she plays.

For any measurable space $\langle \Omega, \Sigma_\Omega\rangle$, we consider $\langle \mathcal{C}(\Omega), \Sigma_{\mathcal{C}(\Omega)}\rangle$ as a measurable space for which the $\sigma$-algebra $\Sigma_{\mathcal{C}(\Omega)}$ is generated by sets $\{\nu \in \mathcal{C}(\Omega) : \nu(E) \geq x\}$ for $E \in \Sigma_\Omega$ and $x \in [0, 1]$. Note that for any event $E \in \Sigma_\Omega$, the set of capacities that believe $E$ is a measurable set in $\Sigma_{\mathcal{C}(\Omega)}$.

For the following exposition, unless noted otherwise, fix a capacity type space $\langle (T_i)_{i\in I}, (s_i)_{i\in I}, (\tau_i)_{i\in I}\rangle$ for a game in strategic form $\langle I, (A_i)_{i\in I}, (u_i \circ o_i)_{i\in I}\rangle$.

Type $t_i$'s conjecture over $A_{-i}$ is defined by $\tau_i(t_i)_{|A_{-i}}(E) := \tau_i(t_i)\left((s_{-i})^{-1}(E)\right)$ for any $E \subseteq A_{-i}$. This is well-defined since for any $j \in I$, $s_j$ is measurable.

In light of Proposition 1 (ii), we define:

**Definition 13** *Type $t_i$* believes *the event $E \in \Sigma_{T_{-i}}$ if $\tau_i(t_i)((T_{-i} \setminus E) \cup F) = \tau_i(t_i)(F)$ for all events $F \in \Sigma_{T_{-i}}$, $F \subseteq E$.*

Next we define Choquet rationality and level-$k$ mutual belief in Choquet rationality as well as Choquet rationality and common belief in Choquet rationality.

**Definition 14** *For $i \in I$ and $k \geq 1$, define inductively,*

$$\begin{aligned} B^1CR_i &= \left\{ t_i \in T_i : \begin{array}{l} s_i(t_i) \text{ is a Choquet best} \\ \text{response to } \tau_i(t_i)_{|A_{-i}} \end{array} \right\} \\ B^{k+1}CR_i &= \left\{ t_i \in B^kCR_i : t_i \text{ believes } B^kCR_{-i} \right\} \end{aligned}$$

*The set of player $i$'s types that satisfy Choquet rationality and common belief in Choquet rationality is*

$$CBCR_i = \bigcap_{k=1}^{\infty} B^kCR_i$$

**Remark 4** *Note that for any player $i \in I$, the set $B^1CR_i$ is measurable because both $s_i$ and $\tau_i$ are measurable. Further, for any measurable set $E \in \Sigma_{T_{-i}}$, $\{t_i \in T_i : t_i \text{ believes } E\}$ is measurable. Thus, for any $k \geq 1$, $B^{k+1}CR_i$ is measurable and $CBCR_i$ is measurable.*

Note that since the notion of Choquet rationality and common belief in Choquet rationality (Definition 20) consists of a conjunction of events, lower continuity of capacities would insure that it is well-defined.

Compared to the definition of Choquet rationalizability, Definition 20 is an epistemic (or better doxastic) notion as it is stated at the level of types that capture player's beliefs about other players. Characterizing both notions in terms of the other would provide an epistemic foundation for Choquet rationalizability in terms of Choquet rationality and common belief in Choquet rationality. That is, we seek to show that any type satisfying Choquet rationality and common belief in Choquet rationality takes a Choquet rationalizability action and for any Choquet rationalizable action there exists a type satisfying Choquet rationality and common belief in Choquet rationality that takes this action. Of course, this epistemic characterization would be relative to the type space. It pertains only to beliefs captured by some type in the type space. A characterization obtained in a particular type space may fail to hold in a different type space. Thus, it is desirable to provide such an epistemic characterization in rich type spaces.

## 4.1 Rich Type Spaces

One notion of rich type space used in the epistemic literature is beliefs-completeness (e.g., Brandenburger, Friedenberg, and Keisler, 2008). Roughly, a type space is beliefs-complete if every belief is captured by some type. Formally, in the context of capacity type-spaces this amounts to requiring that for each player $i \in I$ we have that the type mapping $\tau_i : T_i \longrightarrow \mathcal{C}(T_{-i})$ is a surjection. Unfortunately, in the Appendix C we show that beliefs-complete capacity type spaces do not exist for the general set of capacities.[15] We show this using Cantor's diagonal arguments very much in the spirit of Brandenburger's (2003) impossibility result for possibility structures. This is not surprising because beliefs-completeness of probabilistic type spaces relies on countable additivity of probability measures, which imply continuity of probability measures. General capacities are not necessarily additive and lack any continuity properties without further assumptions. This underlies the importance of continuity of capacities for the epistemic characterization.

In order to facilitate modeling rich spaces that capture beliefs about beliefs about etc. in a setting of measurable spaces, we impose a stronger continuity assumption on capacities that is satisfied automatically in the finite case. Formally, from now on, for any measurable space $(\Omega, \Sigma_\Omega)$ with $\sigma$-algebra $\Sigma_\Omega$, denote now by $\mathcal{C}(\Omega)$ the set of *continuous capacities* on $\Omega$. A capacity $\nu : \Sigma_\Omega \to \mathbb{R}$ is *continuous* if for any increasing (resp. decreasing) sequence of measurable sets $\{E_n\}$, $E_n \in \Sigma_\Omega$ for $n = 1, 2, ...$, with $E_1 \subseteq E_2 \subseteq ...$ (resp. $E_1 \supseteq E_2 \supseteq ...$) and $\bigcup_n E_n = E$ (resp. $\bigcap_n E_n = E$), we have $\lim_{n \to \infty} \nu(E_n) = \nu(E)$. Again, we view continuity of capacities as a technical assumption. While it is possible to characterize it in terms of the underlying Choquet expected utility

[15] This result is also from an earlier unfinished project of Amanda Friedenberg and Burkhard.

preference, it is essentially impossible to test behaviorally. Thus, it makes sense to just state it at the level of capacities.

Further, we let $\mathcal{F}(\Omega)$ denote now the set of all bounded $\Sigma_\Omega$-measurable functions from $\Omega$ to $\mathbb{R}_+$.

Heifetz and Samet (1998) show the existence of the universal type space when beliefs are countable additive probability measures and the space of underlying uncertainties is a measurable space. They point out though that the crucial property in their proof is continuity of probability measures rather than countable additivity per se. This leads them to claim that their construction "can be read now, verbatim, as a proof of the existence of a universal type space, when belief is represented by monotonic, continuous set functions" (Heifetz and Samet, 1998, p. 339, see also Ganguli, Heifetz, and Lee, 2016, Fn. 12). When doing so, we realized two caveats. First, in some parts of their construction they use the uniqueness of the Dirac measure. Yet, a capacity that assigns 1 to a state is not unique. Second, type mappings preserve belief operators that model belief in the context of probabilistic beliefs. Yet, as we pointed out in earlier sections the notion of belief under Choquet expected utility is more subtle. For those reasons we do not see that their construction proves "verbatim" the existence of a universal capacity type space. Therefore, we adopt an alternative approach and derive the universal capacity type space from the universal CEU-representation type space in Appendix D.

Continuous capacities allow for monotone continuous Choquet representations as proved in Appendix. Applying results on the existence of universal representation type spaces in the measurable case for general monotone continuous representations from Heifetz, Ganguli, and Lee (2016) allow us to show the existence of the universal CEU-representation type space (see Appendix for details). In a second step, we map the structure of the collection of CEU-representation type spaces and type morphisms to the collection of continuous capacity type spaces and type morphisms using ideas from category theory. This allows us to claim the existence of the universal capacity type space in the case of measurable spaces and continuous capacities (again, see the Appendix for details). Recently, we also became aware of an unpublished paper by Pinter (2012) who shows the existence of a universal capacity type space for lower continuous capacities in the measurable case.[16]

## 4.2 Epistemic Characterization

The following analysis takes place in the universal capacity type space. We characterize Choquet rationalizability by Choquet rationality and common belief in Choquet rationality. In doing so, we also prove that for every level $k = 1, ...,$ level-$k$ Choquet rationalizability is captured by Choquet rationality and level-$k$-mutual belief in Choquet

[16]Related work is on rich type spaces that allow for ambiguity Epstein and Wang (1996) (for CEU-representation type spaces in the topological case), Ahn (2007) (for sets of countable additive probability measures), and Di Tillo (2008) (for possibly incomplete preferences type spaces).

rationality.

**Theorem 6** *For $i \in I$, $k = 1, ...,$ $R_i^k = s_i(B^k CR_i)$. Moreover, $R_i^\infty = s_i(CBCR_i)$.*

One may be tempted to view the epistemic characterization of Theorem 6 with a caveat since we use the universal capacity type space in the measurable case. Heifetz and Samet (1999) showed by example that the universal probability type space in the measurable case may not contain all coherent hierarchies of beliefs. That is, there are some coherent hierarchies of beliefs that cannot be uniquely extended to a countable additive probability measure on the type space.[17] Since probability measures are a special case of capacities, we would expect a similar impossibility result to apply to the universal continuous capacity type space in the measurable case. It could mean that our universal capacity type space may not capture all hierarchies of beliefs. Yet, from a decision theoretic point of view this may not be as problematic as it may look at the first glance. Recall that we are interested in modeling players that are Choquet expected utility maximizers. Choquet expected utility delivers a capacity, not a sequence of capacities. If there is a hierarchy of capacity beliefs that cannot be expressed as a belief on the type space, then essentially such a hierarchy is not applicable to a Choquet expected utility maximizer on the universal capacity type space. That is, when considering a Choquet expected utility maximizer on the universal capacity type space, only hierarchies of beliefs that can be "pulled" out are relevant.

For probabilistic type spaces, the problem is avoided by assuming familiar topological spaces such as compact Hausdorff (Mertens and Zamir, 1985), Hausdorff (Heifetz, 1993) or Polish (Brandenburger and Dekel, 1994). In such cases, aforementioned have shown that the universal probability type spaces consists of all coherent belief hierarchies. We think that the "topological program" could also be carried out in the case of Choquet expected utility. In fact, a construction very similar to ours in the appendix but using Epstein and Wang (1996) in stead of Ganguli, Heifetz, and Lee (2016) should yield a universal capacity type space in the typological case. In a seminal paper, Epstein and Wang (1996) proved the existence of a universal utility representation type space in the topological case. Their result applies to Choquet expected utility representations.

Previous versions of Theorem 6 appeared in the seminal paper by Epstein (1997) using a representation-based notion of rationalizability (also mentioned in the working paper by Ghirardato and Le Breton, 1999). As mentioned earlier, he considers a rationalizability notion for a general classes of preferences. Although his general approach allows for Choquet expected utility, he did not show the result for Choquet expected utility players in particular and the version of Choquet expected utility required in the context of games had not been developed in 1997. We know from epistemic literature that restrictions on preferences and beliefs may pose challenges for characterizations. Essentially we show

[17]Note that their counterexample features an infinite space of states of nature, while our space of underlying uncertainties is finite (i.e., finite sets of actions).

that restricting his classes of preferences to Choquet expected utility does indeed allow for the characterization. His result is still not a generalization of ours since he considers the topological case while we consider the measurable case. Moreover, instead of a type space commonly considered in game theory he worked with a utility representation type space introduced in Epstein and Wang (1996). Such a representation-based type space is very sensible when working with general preferences. Yet, in order to facilitate comparison with results in the probabilistic case, it is also useful to have a characterization at the level of capacity type spaces. So we view our results as complementary to his.

# 5 Common Belief in Ambiguity Attitudes

The family of Choquet expected utility preferences is a rich model that allows to accommodate ambiguity and players' attitudes towards it. In decision theoretic terms, an individual displays aversion (resp., love) towards ambiguity if she prefers (resp., dislikes) an act that (state-wise) averages utilities of outcomes of two acts to the less favorable act among the two.

To formalize ambiguity attitudes, we use the technique of "*preference averages*" introduced by Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2003).[18] Call an event $E \in \Sigma_\Omega$ *essential* if $x \succ x_E y \succ y$ for some $x, y \in X$. An act $x_E y$ that returns an outcome $x$ on $E$ and $y$ on $\Omega \setminus E$ is called a *bet.* The certainty equivalent of $x_E z$ is denoted by $c(x_E z) \in X$ and defined by $x_E z \sim c(x_E z)$.

We define as in Ghirardato et al. (2003):

**Definition 15** *Let $E$ be an essential event. Given $x, y \in X$, if $x \succsim y$ we say that a consequence $z \in X$ is a preference average of $x$ and $y$ (given $E$) if $x \succsim z \succsim y$ and*

$$x_E y \sim \left(c(x_E z)\right)_E \left(c(z_E y)\right).$$

*If $x \precsim y$, $z$ is said to be a preference average of $x$ and $y$ if it is a preference average of $y$ and $x$.*

"Subjective mixtures" of acts can be defined state-wise. For each $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$, define $\alpha f \oplus (1 - \alpha) g$ to be the act that returns $\alpha f(\omega) \oplus (1 - \alpha) g(\omega) = z$ in state $\omega \in \Omega$ where $z$ satisfies

$$f(\omega)_E g(\omega) \sim \left(c(f(s)_E z)\right)_E \left(c(g(s)_E z)\right)$$

for some essential event $E$.

[18] Recall that we do not have Anscombe-Aumann acts available in our game theoretic framework. Ghirardato et al. (2003) use the technique of preference averages to axiomatize the Choquet expected utility model in the the Savage-style framework. In their words (p. 1898), "*subjective mixtures enable us to readily extend the AA-style axiomatics and techniques to a fully subjective environment*".

Now, we can define the standard notions of ambiguity attitudes using subjective mixtures:

**Definition 16 (Ambiguity Attitudes)** *Let $\succsim$ be a preference relation on $\mathcal{F}$. For any $f, g \in \mathcal{F}$ and any $\alpha \in (0, 1]$, the preference relation $\succsim$ is ambiguity averse if,*

$$f \sim g \text{ implies } \alpha f \oplus (1-\alpha) g \succsim f.$$

*The preference relation $\succsim$ is ambiguity loving if*

$$f \sim g \text{ implies } \alpha f \oplus (1-\alpha) g \precsim f.$$

Notice that a preference relation $\succsim$ is ambiguity neutral (i.e., it coincides with the SEU form) if $\succsim$ is both ambiguity averse and loving. Attitudes towards ambiguity revealed by Choquet preferences are closely related to the form of capacities.

**Definition 17 (Convex/concave capacity)** *A capacity $\nu : \Sigma_\Omega \longrightarrow \mathbb{R}$ is said to be*

*(i) convex, if $\nu(E) + \nu(F) \leq \nu(E \cup F) + \nu(E \cap F)$; and*

*(ii) concave, if $\nu(E) + \nu(F) \geq \nu(E \cup F) + \nu(E \cap F)$,*

*for all events $E, F \in \Sigma_\Omega$.*

The following result is analogous to Schmeidler (1989) and Wakker (1990). Ambiguity aversion (resp., love) is characterized by convex (resp., concave) capacities.[19]

**Proposition 5** *Let $\succsim$ be a Choquet expected utility preference relation on $\mathcal{F}$ with respect to a capacity $\nu$. Then, $\succsim$ is ambiguity averse (resp., loving) if and only if $\nu$ is convex (resp., concave).*

This "prelude" on ambiguity attitudes allows us now study rational behavior under common belief in Choquet rationality and further restrictions imposed on player's ambiguity attitude. In particular, we are interested in comparing the set of rationalizable actions that are feasible under common belief in Choquet rationality and common belief in ambiguity aversion (resp. ambiguity love) with the set of rationalizable actions under common belief in ambiguity neutrality (i.e., subjective expected utility).

[19] There are other notions of ambiguity aversion in the context of Choquet expected utility. For instance, Epstein (1999) attributes ambiguity aversion to *balanced* capacities. A capacity is balanced if its core is a nonempty set. Every convex capacity is balanced, but not *vice versa*. However, for the analysis of rationalizable behavior under common belief Choquet rationality and ambiguity aversion, assuming ambiguity aversion in a weaker sense does not matter.

## 5.1 Choquet Rationalizability with Restrictions on Ambiguity Attitudes

Let $\mathcal{C}^{r_i}(A_{-i})$ be the set of all capacities that satisfy restriction $r_i \subseteq \{conv, conc, add\}$ that stand for concave, convex, and additive capacities (i.e., probabilities), respectively.

Define restricted Choquet rationalizability (henceforth, the $r$-Choquet rationalizability) as follows:

**Definition 18 ($r$-Choquet rationalizability)** *For $i \in I$, $r = (r_i)_{i \in I}$ with $r_i \in \{conv, conc, add\}$, and $k \geq 1$ define inductively,*

$$\begin{aligned}
C_i^{r,1} &= \mathcal{C}^{r_i}(A_{-i}) \\
R_i^{r,1} &= \left\{ a_i \in A_i : \begin{array}{l} \text{there exists } \nu_i \in C_i^{r,1} \text{ for which } a_i \\ \text{is a Choquet best response} \end{array} \right\} \\
&\vdots \\
C_i^{r,k+1} &= \left\{ \nu_i \in C_i^{r,k} : \begin{array}{l} \nu_i((A_{-i} \setminus R_{-i}^k) \cup F) = \nu_i(F) \\ \text{for all } F \subseteq R_{-i}^k \end{array} \right\} \\
R_i^{r,k+1} &= \left\{ a_i \in A_i : \begin{array}{l} \text{there exists } \nu_i \in C_i^{r,k+1} \text{ for which} \\ a_i \text{ is a Choquet best response} \end{array} \right\}
\end{aligned}$$

*The set of $r$-Choquet rationalizable actions is*

$$R_i^{r,\infty} = \bigcap_{k=1}^{\infty} R_i^{r,k}.$$

Although we allow players to display different ambiguity attitudes in the above definition, we are mainly interested in "symmetric" restrictions in which $r_i = r_j$ for all $i, j \in I$. In such a case, we simply write $r = conv$, $r = conc$, or $r = add$. *add*-Choquet rationalizability is just standard rationalizability à la Bernheim (1984) and Pearce (1984).

As the next example demonstrates, the set of Choquet rationalizable actions might expand under ambiguity aversion as compared to the set of rationalizable actions à la Bernheim (1984) and Pearce (1984).

**Example 2 (Coarsening under Ambiguity Aversion)** *Consider the game of Example 1. When both players are ambiguity neutral (i.e., $r = add$), it is easy to verify that*

$$R_1^{add,\infty} = \{u, d\} \text{ and } R_2^{add,\infty} = \{l, r\}.$$

*This is the case of standard rationalizability à la Bernheim (1984) and Pearce (1984). For Rowena, $u$ is first-level rationalizable with a probabilistic belief puts probability larger equal than $\frac{1}{2}$ that Colin plays $l$. Similarly, $d$ is first-level rationalizabile with respect to* *a belief that puts probability larger equal than $\frac{1}{2}$ that $r$ is played. However, there is no probability measure that rationalize choosing $m$. For Colin, $a$ is first-level rationalizable with a belief that assigns sufficiently high probability to Rowena's action $u$, wheres $b$ is first-level rationalizable with belief that assigns sufficiently high probability to action $d$. The same arguments apply for any level $k \geq 2$.*

*Now suppose that Rowena is ambiguity averse with a convex capacity over $\{a, b\}$. Since convexity contains the additive case, $u$ and $d$ are rationalizable at the first level. Furthermore, convex capacities rationalize playing $m$. In particular, for any capacity $\nu_1$ such that $\nu_1(a), \nu_1(b) \in [0, \frac{1}{2})$, actions $m$ is Rowena's best response. Since the reasoning repeats at any level $k \geq 2$, the set of Choquet rationalizable actions under convexity is*

$$R_1^{conv,\infty} = \{u, d, m\} \quad \text{and} \quad R_2^{conv,\infty} = \{l, r\}.$$

Battigalli et al. (2016) present a similar example using the smooth ambiguity model.

However, the set of Choquet rationalizable actions under ambiguity love coincides always with the set of rationalizable actions under additivity.

**Proposition 6** *For each player $i \in I$, $R_i^{conc,\infty} = R_i^{add,\infty} \subseteq R_i^{conv,\infty}$.*

## 5.2 Epistemic Characterization

We define the events in the type space that correspond to player's attitudes towards ambiguity as motivated by Proposition 5.

**Definition 19** *Fix a capacity type space $\langle (T_i), (\tau_i), (s_i) \rangle$. The set of player $i$'s ambiguity averse (resp. loving) types is, respectively, defined by*

$$\begin{aligned} [AA]_i &= \{t_i \in T_i : \tau_i(t_i)_{|A_{-i}} \text{ is convex } \} \\ [AL]_i &= \{t_i \in T_i : \tau_i(t_i)_{|A_{-i}} \text{ is concave } \}. \end{aligned}$$

**Remark 5** *For any player $i \in I$, $[AA]_i$ and $[AL]_i$ are measurable in $T_i$.*

Next, we define common belief in Choquet rationality and common belief in an event $E = \cap_{i \in I} E_i$ where for each player $i \in I$, $E_i$ refers to $i$'s attitude towards ambiguity.

**Definition 20** *For $i \in I$ and $k \geq 1$, define inductively,*

$$\begin{aligned} B^1 CRE_i &= \left\{ t_i \in T_i : \begin{array}{l} s_i(t_i) \text{ is a Choquet best} \\ \text{response to } \tau_i(t_i)_{|A_{-i}} \end{array} \right\} \cap E_i \\ B^{k+1} CRE_i &= \left\{ t_i \in B^k CRE_i : t_i \text{ believes } B^k CRE_{-i} \right\} \end{aligned}$$

*The set of player i's types that satisfy Choquet rationality and common belief in Choquet rationality and the event* $E = (E_i)_{i \in I}$ *is*

$$CBCRE_i = \bigcap_{k=1}^{\infty} B^k CRE_i$$

*where* $E_i = [AA]_i$ *or* $E_i = [AL]_i$*.*

We use acronyms $B^k CRAA$ and $CBCRAA$ for $k$-mutual belief in Choquet rationality, ambiguity aversion, and common belief in Choquet rationality, ambiguity aversion, respectively. Analogously, we use acronyms $B^k CRAL$ and $CBCRAL$ for the "ambiguity love" counterparts.

The following analysis takes places in the universal capacity type space:

**Conjecture 1** *For* $i \in I$ *and* $k = 1, ...,$

*(i)* $s_i(B^k CRAA_i) = R_i^{conv,k}$ *and* $s_i(CBCRAA_i) = R_i^{conv,\infty}$*,*

*(ii)* $s_i(B^k CRAL_i) = R_i^{conc,k}$ *and* $s_i(CBCRAL) = R_i^{conc,\infty}$*.*

# 6 Common Belief in "Strict" Ambiguity Attitudes

Both ambiguity aversion or ambiguity love encompass ambiguity neutrality as a special case. We are also interested in behavior that is rationalizable under genuine "strategic" ambiguity. Player's ambiguity attitudes are mute unless they perceive ambiguity. We say that player $i$ perceives genuine "strategic" ambiguity about her opponents' strategic behavior if her beliefs on the algebra of action profiles $\Sigma_{A_{-i}}$ are non-additive, i.e., there are, at least, two disjoint sets $E, F \subseteq A_{-i}$ for which $\nu_i(E) + \nu_i(F) \neq \nu_i(E \cup F)$. This definition makes sense when ambiguity attitudes are assumed.

**Remark 6** *Given Choquet expected utility preference* $\succeq$ *w.r.t. to capacity* $\nu$*, if* $\nu$ *is convex or concave, then* $\nu$ *satisfies non-additivity if and only if it does not satisfy additive-separability as stated in Proposition 2.*

To explore $r$-Choquet rationalizability for $r = (conv, na)$ or $r = (conc, na)$, where $na$ stands for non-additivity, let us consider again the prior example. Clearly, each action of Rowena is Choquet rationalizable with respect to a convex and non-additive capacity. That is, $R_1^{(conv,na),\infty} = \{u, d, m\}$, showing that the set of Choquet rationalizable actions under ambiguity aversion and strategic ambiguity coarser than under rationalizability à la Bernheim (1984) and Pearce (1984) with probabilistic beliefs.

Previously we observed that Choquet rationalizability with ambiguity love would not refine rationalizability (Proposition 6). This changes when we restrict to the set of non-additive capacities only. The next example demonstrates that Choquet rationalizability under ambiguity love *and* strategic ambiguity refines the set of rationalizability actions à la Bernheim (1984) and Pearce (1984) with probabilistic beliefs.

**Example 3 (Refinement under "Strict" Ambiguity Love)** *Consider the following game:*

| | | *Colin* $l$ | *Colin* $r$ |
|---|---|---|---|
| | $u$ | $4,0$ | $0,4$ |
| *Rowena* | $d$ | $0,4$ | $4,0$ |
| | $m$ | $2,1$ | $2,1$ |

*Rowena's actions $u$ and $d$ are first-level rationalizable with probabilities that assign a sufficiently large mass to action $l$ and $r$, respectively. Moreover, action $m$ is rationalizable with a uniform probability measure over $\{l, r\}$. Thus, when both players are ambiguity neutral,*

$$R_1^{add,\infty} = \{u, m, d\} \text{ and } R_2^{add,\infty} = \{l, r\}.$$

*Now suppose that both players are ambiguity loving with a concave and non-additive capacity, i.e., $r = (conc, na)$. For Rowena, there is no such a capacity that would rationalize $m$ at the first level. Thus, $R_1^{(conc,na),1} = \{u, d\}$ and $C_1^{(conc,na),1} = \{\nu_1 \in \mathcal{C}^{(conc,na)}(\{r, l\}) \mid \nu_1(r), \nu_1(l) \in (\frac{1}{2}, 1]\}$. For Colin, $R_2^{(conc,na),1} = \{r, l\}$ and $C_2^{(conc,na),1} = \mathcal{C}^{(conc,na)}(\{u, d, m\})$. At the second level, $R_1^{(conc,na),2} = \{u, d\}$ and $R_2^{(conc,na),2} = \{l, r\}$, and so on. Therefore,*

$$R_1^{(conc,na),\infty} = \{u, d\} \quad \text{and} \quad R_2^{(conc,na),\infty} = \{l, r\}.$$

The examples demonstrate that Choquet rationalizability with ambiguity aversion (resp., love) together with strategic ambiguity may yield coarser (resp., finer) sets of actions than the than under rationalizability à la Bernheim (1984) and Pearce (1984) with probabilistic beliefs. Yet, this is not generally the case. In particular, the notion of $r$-Choquet rationalizability when $r = (conv, na)$ or $r = (conc, na)$ is highly deficient as the next example demonstrates.

**Example 4 (Non-Existence under "Strict" Ambiguity Aversion)** *The purpose of this example is to show that Choquet rationalizability with ambiguity aversion and strategic ambiguity may be empty in some games.*

*Consider the following game:*

| | | Colin $l$ | $r$ |
|---|---|---|---|
| | $u$ | $4,0$ | $0,4$ |
| Rowena | $d$ | $0,4$ | $4,0$ |
| | $m$ | $4,2$ | $4,1$ |

*When both players are ambiguity neutral, then* $R_1^{(add),\infty} = \{u,d,m\}$ *and* $R_2^{(add),\infty} = \{l,r\}$.

*Now suppose that Rowena is ambiguity averse with respect to a convex and non-additive capacity on* $\{l,r\}$. *At the first level, there is no capacity in* $\mathcal{C}_1^{(conv,na),1}(\{l,r\})$ *that could rationalize playing u and d, respectively. Thus,* $R_1^{(conv,na),1} = \{m\}$. *At the first level, both Colin's actions are rationalizable with respect to convex and non-additive capacities, i.e.,* $R_2^{(conv,na),1} = \{l,r\}$. *At the second level,* $R_1^{(conv,na),1} = \{m\}$ *and* $C_1^{(conv,na),2} = \mathcal{C}_1^{(conv,na)}(\{l,r\})$ *for Rowena. However, at the second level, Colin who has a convex and non-additive capacity can not believe that Rowena is Choquet rational, thus*

$$R_2^{(conv,na),\infty} = \{\emptyset\}.$$

The non-existence example with ambiguity aversion emphasizes that non-existence is not due to "too much refinement" because as we have seen previously, restricting to ambiguity aversion leads to a coarsening of rationalizability. Of course, a similar non-existence example can be presented for the case of ambiguity love and non-additivity.

**Example 5 (Non-Existence under "Strict" Ambiguity Love)** *The purpose of this example is to show that also Choquet rationalizability with ambiguity love and strategic ambiguity may be empty in some games.*

*Consider the following game:*

| | | Colin $l$ | $r$ |
|---|---|---|---|
| | $u$ | $4,1$ | $0,2$ |
| Rowena | $d$ | $0,1$ | $4,2$ |
| | $m$ | $2,2$ | $2,1$ |

*Now suppose that Rowena is ambiguity loving with respect to a convex and non-additive capacity over Colin's actions* $\{l,r\}$. *At the first level, there is no such a capacity that would rationalize playing m. Thus,* $R_1^{(conc,na),1} = \{u,d\}$ *and* $C_1^{(conc,na),1} = \{\nu \in \mathcal{C}^{(conc,na)}(\{l,r\}) \mid \nu(a), \nu(b) \in (\frac{1}{2}, 1]\}$. *For Colin,* $R_2^{(conc,na),1} = \{l,r\}$ *and* $C_2^{(conc,na),1} = \mathcal{C}^{(conc,na)}(\{u,d,m\})$. *At the second level,* $R_1^{(conc,na),2} = \{u,d\}$ *and* $C_1^{(conc,na),2} = C_1^{(conc,na),1}$. *For Colin,* $R_2^{(conc,na),2} = \{r\}$ *for any concave capacity on* $\{u,d\}$, *since r strictly dominates* *l. However, at the third level, Rowena who has a concave and non-additive capacity can not believe that Colin is Choquet rational, thus leading us to*

$$R_1^{(conc,na),\infty} = \{\emptyset\}.$$

These examples demonstrate that a Choquet expected utility maximizer with non-additive beliefs may be incapable of believing that her opponents play a "single" action profile, even though all strategic uncertainty could be eliminated. Therefore, requiring beliefs to be non-additive at any level of reasoning may be unnatural as it assumes that strategic ambiguity can never be "resolved" at some level.[20]

Recall that an event is believed if and only if it is an unambiguous event with the capacity value of 1 (see Proposition 3). It is thus not surprising that a singleton is believed if and only of the Choquet expected utility preference is subjective expected utility preference with respect to a degenerate probability measure. Whenever a singleton is believed, there is no uncertainty whatsoever.

**Corollary 1** *Fix a player $i \in I$ and let $\succsim_i$ her Choquet expected utility preference. Suppose that at some level $k$, the opponents' set of rationalizable actions is a singleton set, i.e., $R_{-i}^{\infty} = \{a_{-i}\}$. Then, player $i$ believes $R_{-i}^{\infty}$ if and only if $\succsim_i$ is a subjective expected utility preference.*

Above examples demonstrate that excluding certainty of a singleton is unnatural. As remedy, we also allow capacities to be Dirac measures. For any player $i \in I$, let $D(A_{-i})$ denote the set of all Dirac measures on $A_{-i}$. Let $\mathcal{C}^{((conv,na)\vee d)}(A_{-i}) = \mathcal{C}^{(conv,na)} \cup D(A_{-i})$ and $\mathcal{C}^{((conc,na)\vee d)}(A_{-i}) = \mathcal{C}^{(conc,na)}(A_{-i}) \cup D(A_{-i})$. So players feel either genuine ambiguity or are certain of just one action profile of opponents.

Now consider $r$-Choquet rationalizability for $r = (conv, na) \vee d$ or $r = (conc, na) \vee d$. Clearly, non-existence vanishes in Examples 4 and 5 when we consider $r$-Choquet rationalizability with either of these two restrictions. Moreover, in Example 3 it still the case that $R_i^{(conv,na)\vee d,\infty}$ is a strict refinement of $R_i^{add,\infty}$ for $i = 1, 2$. These examples demonstrate that any of the following subset relations may be strict for some games.

**Conjecture 2** *For each player $i \in I$, $\emptyset \neq R_i^{(conc,na)\vee d,\infty} \subseteq R_i^{add,\infty} \subseteq R_i^{(conv,na)\vee d,\infty}$.*

## 6.1 Epistemic Characterization

Formally, strategic ambiguity at the level of type is expressed as follows.

[20] The problem is somewhat reminiscent of iterated admissibility. Samuelson (1992) pointed out that while iterated admissibility implicitly models cautious beliefs of players, at higher levels players must somehow uncautiously believe that others do not play actions that cannot be rationalized with a cautious belief. This sparked a large literature on epistemic characterization of iterated admissibility, see Brandenburger, Friedenberg, and Keisler (2008) for the seminal paper.

**Definition 21** *Type $t_i$ faces strategic ambiguity if there are subsets of opponents' action profiles that are ambiguous according to $t_i$'s conjecture over $A_{-i}$. That is, there are disjoint subsets $E, F \subset A_{-i}$ such that $\tau_i(t_i)_{|A_{-i}}(E) + \tau_i(t_i)_{|A_{-i}}(F) \neq \tau_i(t_i)_{|A_{-i}}(E \cup F)$. The event that player $i$ faces strategic ambiguity is denoted by*

$$[SA]_i := \{t_i \in T_i : t_i \text{ faces strategic ambiguity } \}.$$

Note that this definition just requires an ambiguous set of opponents' action profiles. Hence, the name "strategic ambiguity".[21]

**Remark 7** *For any player $i \in I$, $[SA]_i$ is measurable in $T_i$.*

Strategic certainty at the level of type is expressed as follows.

**Definition 22** *Type $t_i$ is strategically certain if there exists $a_{-i} \in A_{-i}$ such that $\tau_i(t_i)_{|A_{-i}}(a_{-i}) = \delta_{a_{-i}}$, where $\delta_{a_{-i}}$ is the Dirac measure on $A_{-i}$ that assigns probability 1 to $a_{-i}$. The event that player $i$ faces strategic certainty is denoted by*

$$[SC]_i := \{t_i \in T_i : t_i \text{ faces strategic certainty } \}.$$

For $i, j \in I$ and $k \geq 1$, we use ("strict ambiguity ...")

$$\begin{aligned}
B^kCRSAA_i &= B^kCRE_i \text{ for } E_j = ([AA]_j \cap [SA]_j) \cup [SC]_j \\
B^kCRSAL_i &= B^kCRE_i \text{ for } E_j = ([AL]_j \cap [SA]_j) \cup [SC]_j \\
CBCRSAA_i &= CBCRE_i \text{ for } E_j = ([AA]_j \cap [SA]_j) \cup [SC]_j \\
CBCRSAL_i &= CBCRE_i \text{ for } E_j = ([AL]_j \cap [SA]_j) \cup [SC]_j
\end{aligned}$$

The following analysis takes places in the universal capacity type space:

**Conjecture 3** *For $i \in I$ and $k = 1, ...,$*

*(i)* $s_i(B^kCRSAA_i) = R_i^{(conv,na)\vee d,k}$ *and* $s_i(CBCRSAA_i) = R_i^{(conv,na)\vee d,\infty}$,

*(ii)* $s_i(B^kCRSAL_i) = R_i^{(conc,na)\vee d,k}$ *and* $s_i(CBCRSAL_i) = R_i^{(conc,na)\vee d,\infty}$.

[21] This terminology is reminiscent of the notion of "strategic assumption" in Heifetz, Meier, and Schipper (2019) in which only the marginal of the lexicographic belief system on opponents' actions is required to have full support but may not have full support with respect to opponents' types.

# 7 Discussion

## 7.1 Related Literature

# A Proofs and Additional Material of Section 2

## A.1 Proof of Proposition 1

`Adam: Update to new representation used.`

We begin with two lemmata on null.

**Lemma 6** *Fix a null event $E \in \Sigma_\Omega$. Then, for all events $F, G \in \Sigma_\Omega$, $F \subseteq E$ and $G \subseteq \Omega$, $\nu(F \cup G) = \nu(G)$.*

Proof. Suppose $E \in \Sigma_\Omega$ is null. Fix events $F, G \in \Sigma_\Omega$, $F \subseteq E$ and $G \subseteq \Omega$. Since $G \subseteq F \cup G$, $\nu(G) \leq \nu(F \cup G)$. Suppose $\nu(G) < \nu(F \cup G)$. Let

$$f(\omega) = \begin{cases} 1 & \text{if } \omega \in F \cup (G \cap E) \\ 0 & \text{otherwise,} \end{cases}$$
$$g(\omega) = \begin{cases} 1 & \text{if } \omega \in G \cap E \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h(\omega) = \begin{cases} 1 & \text{if } \omega \in G \cap (\Omega \setminus E) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \int_\Omega f_E h(\omega) d\nu(\omega) &= \nu(F \cup G) \\ &> \nu(G) = \int_\Omega g_E h(\omega) d\nu(\omega), \end{aligned}$$

contradicting that $E$ is null. □

**Lemma 7** *Suppose, for all events $E, F, G \in \Sigma_\Omega$, $F \subseteq E$ and $G \subseteq \Omega$, $\nu(F \cup G) = \nu(G)$. Then $E$ is null.*

Proof. Fix a capacity as in the statement of the Lemma. It suffices to show that, for any acts $f, g, h \in \mathcal{F}$, $f_E h \succsim g_E h$. Notice

$$\begin{aligned} \int_\Omega f_E h(\omega) d\nu(\omega) &= \int_{[0,1]} \nu\left(\{\omega \in E : f(\omega) \geq t\} \cup \{\omega \in \Omega \setminus E : h(\omega) \geq t\}\right) dt \\ &= \int_{[0,1]} \nu\left(\{\omega \in \Omega \setminus E : h(\omega) \geq t\}\right) dt, \end{aligned}$$

where the second line follows from that fact that, for any events $F, G \in \Sigma_\Omega$ with $F \subseteq E$, $\nu(F \cup G) = \nu(G)$. A similar argument establishes that

$$\int_\Omega g_E h(\omega) d\nu(\omega) = \int_{[0,1]} \nu\left(\{\omega \in \Omega \setminus E : h(\omega) \geq t\}\right) dt.$$

So,

$$\int_\Omega f_E h(\omega) d\nu(\omega) = \int_\Omega g_E h(\omega) d\nu(\omega),$$

as required. □

We are now ready to prove Proposition 1.

First, fix an event $E \in \Sigma_\Omega$ that is believed. Then $\Omega \setminus E$ is null. Lemma 6 yields that for all events $G, H \in \Sigma_\Omega$, $H \subseteq \Omega \setminus E$ and $G \subseteq \Omega$, $\nu(H \cup G) = \nu(G)$. So, certainly, for all events $G \in \Sigma_\Omega$, $G \subseteq E$, $\nu((\Omega \setminus E) \cup G) = \nu(G)$.

Next, suppose that $\nu((\Omega \setminus E) \cup F) = \nu(F)$ for all events $F \in \Sigma_\Omega$, $F \subseteq E$. It suffices to show that, for all events $G, H \in \Sigma_\Omega$ with $H \subseteq \Omega \setminus E$ and $G \subseteq \Omega$,

$$\nu(H \cup G) = \nu(G).$$

If so, then the result is immediate from Lemma 7.

Fix an event $G \in \Sigma_\Omega$. Certainly,

$$\nu((\Omega \setminus E) \cup G) = \nu((\Omega \setminus E) \cup (G \cap E)) = \nu(G \cap E), \tag{12}$$

where the first equality follows from the fact that $G \cap (\Omega \setminus E) \subseteq \Omega \setminus E$ and the second equality follows from taking $F = G \cap E$. Now notice that

$$\begin{aligned} \nu((\Omega \setminus E) \cup G) &\geq \nu(G) \\ &\geq \nu(G \cap E) \\ &= \nu((\Omega \setminus E) \cup G), \end{aligned} \tag{13}$$

where the first two lines follow from monotonicity and the third line follows from Equation (12). From Equation (13),

$$\nu(G) = \nu(G \cap E). \tag{14}$$

Taken together, Equations (12) and (14) establish that, for any event $G \subseteq \Omega$, we must have

$$\nu((\Omega \setminus E) \cup G) = \nu(G). \tag{15}$$

Now, fix an event $H \in \Sigma_\Omega$ with $H \subseteq \Omega \setminus E$. Then

$$\begin{aligned} \nu(G) &\leq \nu(H \cup G) \\ &\leq \nu((\Omega \setminus E) \cup G) \\ &= \nu(G), \end{aligned} \tag{16}$$

where the first three lines follow from monotonicity and the last from Equation (15). By Equation (16)

$$\nu(H \cup G) = \nu(G), \tag{17}$$

as required. □

## A.2 Proof of Proposition 2

`TBA by Adam`

## A.3 Examples on Additivity versus Unambiguous Events

**Example 6** *The purpose of the example is to show that additivity of a capacity w.r.t. to an event does not imply that the event is perceived unambiguous.*

*Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Consider the following capacity $\nu$ defined by*

| | $\emptyset$ | $\{\omega_1\}$ | $\{\omega_2\}$ | $\{\omega_3\}$ | $\{\omega_1, \omega_2\}$ | $\{\omega_1, \omega_3\}$ | $\{\omega_2, \omega_3\}$ | $\Omega$ |
|---|---|---|---|---|---|---|---|---|
| $\nu(\cdot)$ | 0 | $\frac{1}{3}$ | $\frac{1}{6}$ | $\frac{1}{3}$ | 1 | 1 | $\frac{2}{3}$ | 1 |

*The capacity $\nu$ is additive on the partition $\{\{\omega_1\}, \{\omega_2, \omega_3\}\}$. Yet, $\{\omega_1\}$ is ambiguous.*

$$\begin{aligned} \nu(\{\omega_1, \omega_2\}) &> \nu(\{\omega_1, \omega_2\} \cap \{\omega_1\}) + \nu(\{\omega_1, \omega_2\} \cap \{\omega_2, \omega_3\}) \\ \nu(\{\omega_1, \omega_2\}) &> \nu(\{\omega_1\}) + \nu(\{\omega_2\}) \\ 1 &> \frac{1}{3} + \frac{1}{6} = \frac{1}{2}. \end{aligned}$$

**Example 7** *This example is to show that if there exists an ambiguous event w.r.t. to a capacity $\nu$, it does not imply that there exists an event $E \in \mathcal{F}_E$ on which the capacity is non-additive, i.e., $\nu(E) + \nu(\Omega \setminus E) \neq 1$.*

*Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and consider the following capacity $\nu$ defined by*

| | $\emptyset$ | $\{\omega_1\}$ | $\{\omega_2\}$ | $\{\omega_3\}$ | $\{\omega_1, \omega_2\}$ | $\{\omega_1, \omega_3\}$ | $\{\omega_2, \omega_3\}$ | $\Omega$ |
|---|---|---|---|---|---|---|---|---|
| $\nu(\cdot)$ | 0 | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | 1 |

*Capacity $\nu$ is additive on $\{\{\omega\}, \Omega \setminus \{\omega\}\}$ for every $\omega \in \Omega$. Yet, every event is ambiguous. Consider event $\{\omega_1, \omega_2\}$:*

$$\begin{aligned} \nu(\{\omega_1,\omega_2\}) &> \nu(\{\omega_1,\omega_2\} \cap \{\omega_1\}) + \nu(\{\omega_1,\omega_2\} \cap \{\omega_2,\omega_3\}) \\ \nu(\{\omega_1,\omega_2\}) &> \nu(\{\omega_1\}) + \nu(\{\omega_2\}) \\ \frac{3}{4} &> \frac{1}{4} + \frac{1}{4} = \frac{1}{2}. \end{aligned}$$

## Proof of Proposition 3

Throughout the proof fix a Choquet expected utility preference $\succsim$ w.r.t. a capacity $\nu$ on $\Sigma_\Omega$.

"$(i) \Longrightarrow (iii)$": Let $E \in \Sigma_\Omega$ be a $\succsim$-believed event. Thus, it holds true that $\nu(G) = 0$ for all $G \subseteq \Omega \setminus E$. Suppose, to the contrary, that $\nu(G) > 0$ for some $G \subseteq \Omega \setminus E$. Then, by $(iii)$ of Proposition 1, $\nu(G \cup \{\emptyset\}) = \nu(\{\emptyset\}) = 0$, yielding a contradiction. Thus, for any $F \in \Sigma_\Omega$, $\nu(F \cap (\Omega \setminus E)) = 0$, and the following holds true:

$$\begin{aligned} \nu(F) &= \nu\big((F \cap E) \cup (F \cap (\Omega \setminus E))\big) = \nu(F \cap E) \\ &= \nu(F \cap E) + \nu(F \cap (\Omega \setminus E)). \end{aligned}$$

Thus, by Definition 2, event $\Omega \setminus E$ is unambiguous event (see Equation (10)) with $\nu(E) = 0$.

"$(iii) \Longrightarrow (ii)$": By Equation 10, if $\Omega \setminus E$ is unambiguous, so is event $E$ and thus $\nu(E) = 1$.

"$(ii) \Longrightarrow (i)$": Let $E \in \Sigma_\Omega$ be an unambiguous event with $\nu(E) = 1$. Take $F \subseteq E$ for $F \in \Sigma_\Omega$. Thus,

$$\begin{aligned} \nu((\Omega \setminus E) \cup F) &= \nu(((\Omega \setminus E) \cup F) \cap E) + \nu(((\Omega \setminus E) \cup F) \cap (\Omega \setminus E)) \\ &= \nu(F) + \nu(\Omega \setminus E) = \nu(F). \end{aligned}$$

Thus, by $(ii)$ of Proposition 1, event $E$ is believed. □

## A.4 Proof of Proposition 4

Fix a Choquet expected utility preference $\succeq$ w.r.t a capacity $\nu$ on $\Omega$.

Necessity: Trivial.

Monotonicity: Fix an event $E \in \mathcal{F}_\Omega$ that is $\succeq$-believed under $\nu$ and an event $G \in \mathcal{F}_\Omega$ with $E \subseteq G$. By Proposition 1 (ii), since $E$ is $\succeq$-believed under $\nu$,

$$\nu((\Omega \setminus E) \cup F) = \nu(F) \text{ for all events } F \subseteq \Omega.$$

Since $E \subseteq G$, $\Omega \setminus G \subseteq \Omega \setminus E$, so that by monotonicity of $\nu$

$$\nu((\Omega \setminus G) \cup F) \leq \nu((\Omega \setminus E) \cup F) \text{ for all events } F \subseteq \Omega.$$

Again, by monotonicity of $\nu$,

$$\nu((\Omega \setminus G) \cup F) = \nu(F) \text{ for all events } F \subseteq \Omega.$$

Conjunction Part I: Immediate from monotonicity of belief.

Finite Conjunction: Suppose each $E_1, \ldots, E_n$ are $\succeq$-believed under $\nu$.

First notice that $\bigcup_{i=1}^n [\Omega \setminus E_i] = \Omega \setminus \bigcap_{i=1}^n E_i$. Certainly, for any $i$, if $\omega \notin E_i$ then $\omega \notin \bigcap_{j=1}^n E_j$. So, $\bigcup_{i=1}^n [\Omega \setminus E_i] \subseteq \Omega \setminus \bigcap_{i=1}^n E_i$. Conversely, if $\omega \notin \bigcap_{j=1}^n E_j$ then there must exist some $i$ so that $\omega \notin E_i$. From this, it follows that $\Omega \setminus \bigcap_{i=1}^n E_i \subseteq \bigcup_{i=1}^n [\Omega \setminus E_i]$.

Now, by Proposition 1 (ii), for each $j = 1, \ldots, n-1$ and for any event $F$,

$$\nu\left(\bigcup_{i=j}^{n} (\Omega \setminus E_i) \cup F\right) = \nu\left(\bigcup_{i=j+1}^{n} (\Omega \setminus E_i) \cup F\right),$$

and

$$\nu((\Omega \setminus E_n) \cup F) = \nu(F).$$

Taken together

$$\nu\left(\left(\Omega \setminus \bigcap_{i=1}^{n} E_i\right) \cup F\right) = \nu\left(\bigcup_{i=1}^{n} (\Omega \setminus E_i) \cup F\right) = \nu(F),$$

as required.

## A.5 Counterexample Conjunction

**Example 8** *The purpose of this example is to show that events $E_1, E_2, \ldots$ may be $\succeq$-believed under $\nu$, even though $\bigcap_i E_i$ is not $\succeq$-believed under $\nu$.*[22] *(Fix a Choquet expected utility preference $\succeq$ w.r.t. $\nu$.) Let $\Omega = [0,1]$ endowed with the relative topology. For each $i \geq 2$, define $E_i = \left[0, \frac{1}{i}\right]$ and set*

$$\nu(F) = \begin{cases} 1 & \text{if there is some } i \geq 2 \text{ with } E_i \subseteq F \\ 0 & \text{otherwise.} \end{cases}$$

*First note that each $E_i$ is $\succeq$-believed. To see this, fix some event $F_i \subseteq E_i$. Note, if $\nu((\Omega \setminus E_i) \cup F_i) = 1$ then there exists some $j$ with $\left[0, \frac{1}{j}\right] \subseteq (\Omega \setminus E_i) \cup F_i$. If $j \leq i$ then $\left[0, \frac{1}{i}\right] \subseteq \left[0, \frac{1}{j}\right]$ so that*

$$\left[0, \frac{1}{i}\right] \subseteq \left[0, \frac{1}{j}\right] \subseteq (\Omega \setminus \left[0, \frac{1}{i}\right]) \cup F_i$$

[22]The example is due to Amanda Friedenberg.

*or* $\left[0,\frac{1}{i}\right]\subseteq F_i$. *From this,* $\nu(F_i)=1$, *as required. If* $i<j$ *then* $\left[0,\frac{1}{j}\right]\subseteq\left[0,\frac{1}{i}\right]$. *Using this and the fact that* $\left[0,\frac{1}{j}\right]\subseteq\left(\Omega\setminus\left[0,\frac{1}{i}\right]\right)\cup F_i$, *it follows that* $\left[0,\frac{1}{j}\right]\subseteq F_i$ *and so* $\nu(F_i)=1$, *as required. If* $\nu((\Omega\setminus E_i)\cup F_i)=0$ *certainly* $\nu(F_i)=0$.

*Also notice that* $\cap_{i=2}^{\infty}E_i$ *is not* $\succeq$*-believed, since*

$$\begin{aligned}\nu\left((\Omega\setminus\cap_{i=2}^{\infty}E_i)\cup\cap_{i=2}^{\infty}E_i\right) &= 1\\ &> 0=\nu\left(\cap_{i=2}^{\infty}E_i\right).\end{aligned}$$

## A.6 Proof of Lemma 4

First notice that $\Omega\setminus\bigcap_{n=1}^{\infty}E_n=\bigcup_{n=1}^{\infty}(\Omega\setminus E_n)$. To see this, first notice that if $\omega\in\Omega\setminus\bigcap_{n=1}^{\infty}E_n$ then there exists some $n$ with $\omega\in\Omega\setminus E_n$. This establishes that $\omega\in\bigcup_{n=1}^{\infty}(\Omega\setminus E_n)$. Conversely, suppose there exists some $n$ with $\omega\in\Omega\setminus E_n$. Then certainly $\omega\notin\bigcap_{n=1}^{\infty}E_n$, establishing that $\omega\in\Omega\setminus\bigcap_{n=1}^{\infty}E_n$.

Fix an event $F\subseteq\bigcap_{n=1}^{\infty}E_n$. Define $G_n=(\Omega\setminus E_n)\cup F$ and note that $G_n\subseteq G_{n+1}$ since $E_{n+1}\subseteq E_n$. Since, for each $n$, $E_n$ is $\succeq$-believed and $F\subseteq\bigcap_{n=1}^{\infty}E_n\subseteq E_n$, $\nu(G_n)=\nu(F)$. It follows that $\lim_{n\to\infty}\nu(G_n)=\nu(F)$. Now, since $\nu$ is lower continuous,

$$\begin{aligned}\nu(F) &= \lim_{n\to\infty}\nu(G_n)\\ &= \nu\left(\bigcup_{n=1}^{\infty}G_n\right)\\ &= \nu\left(\left(\Omega\setminus\bigcap_{n=1}^{\infty}E_n\right)\cup F\right),\end{aligned}$$

as required.

## A.7 Counterexample of Conjunction for Regular Capacities

The purpose of this section is to demonstrate that replacing lower continuity with regularity may let conjunction fail.[23] We make use of Example 8 and show that it features a 'nice' capacity, as it satisfies a regularity condition:

**Definition 23** *A capacity* $\nu$ *is regular if*

1. *for each event* $E$, $\nu(E)=\sup\{\nu(C):C\subseteq E$ *and* $C$ *closed*$\}$
2. *for each closed set* $C$, $\nu(C)=\inf\{\nu(U):C\subseteq U$ *and* $U$ *open*$\}$.

[23] The example and arguments are due to Amanda Friedenberg.

**Claim 1** *The capacity in Example 8 is a regular capacity.*

First we will show property 1, i.e. for any event $F \subseteq \Omega$, $\nu(F) = \sup\{\nu(C) : C \subseteq F$ and $C$ closed$\}$. Fix an event $F$ with $\nu(F) = 1$. Then there exists some closed $E_i = [0, \frac{1}{i}]$ with $E_i \subseteq F$, i.e. $\nu(F) = \nu(E_i) = 1$ for $E_i \subseteq F$ closed. If $\nu(F) = 0$, then for any closed $C \subseteq F$ (and there must be some) we have $\nu(C) = 0 = \nu(F)$ establishing the requirement.

Next will will show property 2, i.e. for any closed $C \subseteq \Omega$, $\nu(C) = \inf\{\nu(U) : C \subseteq U$ and $U$ open$\}$. Fix a closed set $C$. First suppose that $\nu(C) = 1$ and note that for any open $U$ with $C \subseteq U$ we must have that $\nu(U) = 1$ establishing the requirement.

Now suppose that $\nu(C) = 0$. We must only show that there is some open $U$ with $\nu(U) = 0$ and $C \subseteq U$.

Since $\nu(C) = 0$, for each $i$, $\left[0, \frac{1}{i}\right] \cap ([0,1] \setminus C) \neq \emptyset$. For each $i$, fix exactly one $d_i \in \left[0, \frac{1}{i}\right] \cap ([0,1] \setminus C)$ and let $D$ be the set of $d_1, d_2, \ldots$. Note, that $D$ is closed. To show this, we will show that $D$ contains its accumulation points. Specifically, we will show that if $x \in [0,1] \setminus D$ then there exists an open interval around $x$ whose intersection with $D$ is empty. Without loss of generality, assume $d_1 < x < d_2$ and, for each $i$, either $d_i \leq d_1$ or $d_2 \leq d_i$. Then, for $x - d_1 > \varepsilon > d_2 - x$, $(x - \varepsilon, x + \varepsilon)$ satisfies the said conditions.

Given that $C$ and $D$ are closed in $[0,1]$ they are compact. It follows from Kelly (Theorem 5.9) that there exists disjoint open sets $U, V$ with $C \subseteq U$ and $D \subseteq V$. For each $i$, there is then some $d_i \in V$ and (since $U$ and $V$ are disjoint) $d_i \notin U$. It follows that $\nu(U) = 0$ as required.

## A.8 Relationship between Belief and Support Notions

`Adam, update to new representation`

There are a few known support notions suggested in the context of game theory. In their seminal paper, Dow and Werlang (1994) introduce the following support notion.

**Definition 24** *Let be capacity $\nu$. A event $E \in \Sigma_\Omega$ is called a DW-support of $\nu$ if,*

$$\begin{array}{lll} (i) & \nu(\Omega \setminus E) = 0, & \textit{and} \\ (ii) & \nu(\Omega \setminus F) > 0 & \textit{for any event } F \subset E. \end{array}$$

A DW-support always exists, but it might be non-unique. Denote by $\mathcal{DW}(\nu)$ the set of DW supports associated with $\nu$.

Another support notion has been suggested by Marinacci (2000).

**Definition 25** *Let be capacity $\nu$. The M-support of $\nu$ is defined to be*

$$M := \{\omega \in \Omega \mid \nu(\omega) > 0\}.$$

The Marinacci-support may be an empty set. However, if it exists, it is always unique.

For a given capacity $\nu$, let $M$ be the M-suppport, $D$ a DW-support (i.e., $D \in \mathcal{DW}(\nu)$) and $E$ the believed event. Then, one has:

$$\emptyset \subseteq M \subseteq D \subseteq E \subseteq \Omega. \tag{18}$$

It is also known that the DW-support is unique if and only if it is the Marinacci-support (see Eichberger and Kelsey 2014). In this case,

$$\emptyset \neq M = D \subseteq E \subseteq \Omega. \tag{19}$$

The example below shows that the unique DW-support does not need to be believed.

**Example 9** *Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Consider the following capacity $\nu$ on $2^\Omega$: $\nu(\omega_1) = \nu(\omega_2) = \nu(\omega_1, \omega_2) = 0, \nu(\omega_3) = \nu(\omega_1, \omega_3) = \frac{1}{2}$ and $\nu(\omega_2, \omega_3) = 1$. Hence, $\{\omega_2, \omega_3\}$ is believed. Yet, $\{\omega_3\}$ is the unique DW-support and thus it is also the Marinacci-support.*

Also, a DW-support with the capacity value equal to 1 does not need to be believed.

**Example 10** *Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Consider the following capacity $\nu$ on $2^\Omega$: $\nu(\omega_1) = \nu(\omega_2) = 0$, $\nu(\omega_3) = \frac{1}{3}$ and $\nu(\omega_1, \omega_2) = \nu(\omega_1, \omega_3) = \nu(\omega_2, \omega_3) = 1$. Now, $\{\omega_3\}$ is the M-support while $\{\omega_1, \omega_3\}$ and $\{\omega_2, \omega_3\}$ but not $\Omega$ are the DW-supports. However, the believed event is $\Omega$.*

One has the following relationship.

**Lemma 8** *Let $M \in \Sigma_\Omega$ be the Marinacci-support of a capacity $\nu$, provided it exists. Then, $M$ is believed if, and only if, $M$ is the (unique) unambiguous DW-support.*

Proof. (Sketch) "$\Longrightarrow$" Suppose $M$ is believed. By Lemma 2, $M$ is an unambiguous event and $\nu(\Omega \setminus M) = 0$. Now, take a proper subset $F \subset M$. Since $M$ is believed, one has $\nu(\Omega \setminus F) = \nu((M \setminus F) \cup (\Omega \setminus M)) = \nu(M \setminus F)$. Since $M$ is the M-support, there exists $\omega \in M \setminus F$ with $\nu(\omega) > 0$. By monotonicity, $\nu(M \setminus F) > 0$ . Hence, $M$ is a DW-support. In fact, $M$ is the unique DW-support. Suppose it is not. Then, there exists another DW-support, $D$, which is not the M-support. Hence, there exists $\omega \in D$ with $\nu(\omega) = 0$, i.e., $\omega \in \Omega \setminus M$. Let $F := D \setminus \{s\}$. Since $\nu(\Omega \setminus D) = 0$ and $D$ is unambiguous, one has $\nu(\Omega \setminus F) = \nu(\{\omega\} \cup (\Omega \setminus D)) = \nu(\{\omega\}) + \nu(\Omega \setminus D)) = 0$, contradicting that $D$ is a DW-support. Thus, $M$ is the unique unambiguous DW-support.

"$\Longleftarrow$" Let $D$ be the unique unambiguous DW support. By uniqueness and Proposition (*) in Eichberger and Kelsey (2014), $D$ is the M-support. By Lemma 2 and $\nu(\Omega \setminus D) = 0$, $D$ is believed.

# B Proofs of Section 3

## B.1 Proof of Remark ??

Suppose by contradiction that there exists $a_i \in R_i^{k+1}$ with $a_i \notin R_i^k$. The fact that $a_i \in R_i^{k+1}$ is equivalent to saying that there exists a capacity $\nu_i \in C_i^{k+1}$ for which $a_i$ is a Choquet best response among all actions in $A_i$. The fact that $a_i \notin R_i^k$ is equivalent to saying that there does not exist a capacity $\nu_i' \in C_i^k$ for which $a_i$ is a Choquet best response among all actions in $A_i$. By definition, $C_i^{k+1} \subseteq C_i^k$ for all $i \in I$, a contradiction.

## B.2 Proof of Theorem ??

Fix a finite game in strategic form. Since the game is finite and the procedure is monotone in the sense of Remark **??**, there must exist an integer $\bar{k}$ such that for all $k \geq \bar{k}$, $R_i^k = R_i^{\bar{k}}$ for all $i \in I$. For any player $i \in I$ and any $\nu_i \in \mathcal{C}(A_{-i})$ there exists a Choquet best response. Thus, $R_i^{\bar{k}} \neq \emptyset$ for all $i \in I$. Again, by Remark **??**, $R_i^\infty = R_i^{\bar{k}} \neq \emptyset$ for all $i \in I$.

## B.3 Proof of Remark 1

Consider $a_i \in R_i$. By definition, there exist a capacity $\nu_i^{a_i} \in \mathcal{C}(A_{-i})$ such that $\nu_i^{a_i}((A_{-i} \setminus R_{-i}) \cup F) = \nu_i^{a_i}(F)$ for all $F \subseteq R_{-i}$. That is, $R_{-i}$ is believed with $\nu_i^{a_i}$. Since $R_{-i} \subseteq R_{-i} \cup \tilde{R}_{-i}$, monotonicity of belief (see Proposition 4) implies that also $R_{-i} \cup \tilde{R}_{-i}$ is believed with $\nu_i^{a_i}$. Note that this holds for any action in $R_i \cup \tilde{R}_i$. Thus, for any $a_i' \in R_i \cup \tilde{R}_i$ there is a capacity $\nu_i^{a_i'} \in \mathcal{C}(A_{-i})$ with $\nu_i^{a_i'}((A_{-i} \setminus (R_{-i} \cup \tilde{R}_{-i})) \cup F) = \nu_i^{a_i'}(F)$ for all $F \subseteq R_{-i} \cup \tilde{R}_{-i}$ such that $a_i'$ is a Choquet best response with $\nu_i^{a_i'}$. Finally note that this holds for all $i \in I$. Hence, $R_i = \tilde{R}_i$ for all $i \in I$ since otherwise $R_i$ and $\tilde{R}_i$ cannot be largest sets satisfying the property of Definition 6.

## B.4 Proof of Theorem 2

$R_i^\infty \subseteq R_i$ is immediate from definitions. We prove the converse by induction.

For any player $i \in I$, $a_i \in R_i$ implies that $a_i$ is a Choquet best response with respect to $\nu_i^{a_i} \in \mathcal{C}(A_{-i})$ satisfying $\nu_i^{a_i}((A_{-i} \setminus R_{-i}) \cup F) = \nu_i^{a_i}(F)$ for all $F \subseteq R_{-i}$. This implies that $\nu_i^{a_i} \in C_i^1$. Since this holds for all actions in $R_i$, we have $R_i \subseteq R_i^1$.

Induction hypothesis: $R_i \subseteq R_i^\ell$ for all $i \in I$, $\ell \leq k$.

Induction step: We need to show $R_i \subseteq R_i^{k+1}$ for all $i \in I$. For any $i \in I$ and any $a_i \in R_{-i}$ there exists $\nu_i^{a_i}$ such that $R_{-i}$ is believed and $a_i$ is a Choquet best response to $\nu_i^{a_i}$. By the induction hypothesis and monotonicity of beliefs (Appendix 4) $\nu_i^{a_i}$ also believes $R_{-i}^k$. Thus, $\nu_i^{a_i} \in C_i^{k+1}$. Thus, $a_i \in R_i^{k+1}$.

From Remark **??** and arguments in the proof of Theorem **??** follows that $R_i = R_i^\infty$ for all $i \in I$.

## B.5 Proof of Remark 2

We prove by induction. Clearly, for all $i \in I$, $U_i^1(\mathcal{A}) \subseteq U_i^0(\mathcal{A})$.

Assume now for $k \geq 1$, $U_i^k(\mathcal{A}) \subseteq U_i^{k-1}(\mathcal{A})$ for all $i \in I$. Suppose by contradiction that for $k \geq 1$, $A_i' \in U_i^{k+1}(\mathcal{A})$ and $A_i' \notin U_i^k(\mathcal{A})$.

$A_i' \in U_i^{k+1}(\mathcal{A})$ means that for all $a_i \in A_i'$ there is no $\alpha_i \in \Delta(U_i^k(\mathcal{A}) \cap \mathcal{A}_i^\circ)$ such that $\tilde{u}_i(\alpha_i, A_{-i}') > \tilde{u}_i(\{a_i\}, A_{-i}')$ for all $A_{-i}' \in U_{-i}^k(\mathcal{A})$.

$A_i' \notin U_i^k(\mathcal{A})$ means that there exist $a_i \in A_i'$ and $\alpha_i \in \Delta(U_i^{k-1}(\mathcal{A}) \cap \mathcal{A}_i^\circ)$ such that $\tilde{u}_i(\alpha_i, A_{-i}') > \tilde{u}_i(\{a_i\}, A_{-i}')$ for all $A_{-i}' \in U_{-i}^{k-1}(\mathcal{A})$. Since by the induction hypothesis, $U_i^k(\mathcal{A}) \subseteq U_i^{k-1}(\mathcal{A})$ for all $i \in I$, former statement also holds for all $A_{-i}' \in U_{-i}^k(\mathcal{A})$.

Thus, if $a_i \in A_i'$ for which there exists $\alpha_i \in \Delta(U_i^{k-1}(\mathcal{A}) \cap \mathcal{A}_i^\circ)$ such that $\tilde{u}_i(\alpha_i, A_{-i}') > \tilde{u}_i(\{a_i\}, A_{-i}')$ for all $A_{-i}' \in U_{-i}^k(\mathcal{A})$, then $\alpha_i(\{a_i'\}) > 0$ for some $\{a_i'\} \in (U_i^{k-1}(\mathcal{A}) \setminus U_i^k(\mathcal{A})) \cap \mathcal{A}_i^\circ$. But any such action $\{a_i'\}$ is strictly dominated at level $k$. Thus, we can improve the expected payoff of player $i$ by shifting probability mass away from $\{a_i'\}$ to actions in $U_i^k(\mathcal{A}) \cap \mathcal{A}_i^\circ$. For the resulting mixed action, let's denote it by $\alpha_i'$, we have $\alpha_i' \in \Delta(U_i^k(\mathcal{A}) \cap \mathcal{A}_i^\circ)$ such that $\tilde{u}_i(\alpha_i', A_{-i}') > \tilde{u}_i(\{a_i\}, A_{-i}')$ for all $A_{-i}' \in U_{-i}^k(\mathcal{A})$, a contradiction to $A_i' \in U_i^{k+1}(\mathcal{A})$.

## B.6 Proof of Theorem 3

We prove by induction. For any $i \in I$, let $R_i^0 = A_i$. Then $R_i^0 = A_i^0$ for all $i \in I$. Assume now that for $k \geq 0$, $R_i^k = A_i^k$ for all $i \in I$. We need to show that $R_i^{k+1} = A_i^{k+1}$ for all $i \in I$.

By Remark 3, $a_i \in A_i^{k+1}$ if and only if $\{a_i\} \in U_i^{k+1}(\mathcal{A})$ for any $i \in I$. Note that by definition $U^k(\mathcal{A})$ is an extended restriction (of every player). Further $R_i^k = A_i^k$ if and only if $U_i^k(\mathcal{A}) = 2^{R_i^k} \setminus \{\emptyset\}$ for $i \in I$. Thus, $U^k(\mathcal{A})$ is an extended restriction in the extend game associated with the restriction $R^k$ in the underlying game. Now from Lemma 5 follows that $a_i$ is a Choquet best response given restriction $R^k$ if and only if $\{a_i\}$ is not strictly dominated in the associated extended restriction $U^{k+1}(\mathcal{A})$. Hence for $i \in I$, $a_i \in R_i^{k+1}$ if and only if $\{a_i\} \in U_i^{k+1}(\mathcal{A})$ if and only if (by Remark 3) $a_i \in A_i^{k+1}$.

## B.7 Proof of Theorem 5

For any player $i \in I$ and $k = 0$, simply define $R_i^0 = A_i$.

We prove by induction.

For any $a_i \in R_i^1$ there exists $\nu_i \in C_i^1$ for which $a_i$ is a Choquet best response. Let $\succeq_i$ be the Choquet expected utility preference relation on $\mathcal{F}^{A_i}$ w.r.t. $\nu_i$. Then $R_{-i}^0$ is $\succeq_i$-believed. This follows simply from $R_{-i}^0 = A_{-i}$. Moreover, $f^{a_i} \succeq_i g$ for all $g \in \mathcal{F}^{A_i}$. Let $CEU_i \in \mathcal{R}^{u_i}(A_{-i} \mid E_{-i}^0) = \mathcal{R}^{u_i}(A_{-i} \mid A_{-i}) = \mathcal{R}^{u_i}(A_{-i})$ such that $CEU_i$ represents $\succeq_i$ with $\nu_i$ and $u_i$. Then $a_i \in E_i^1$.

Conversely, for any $a_i \in E_i^1$, there exist a Choquet expected utility function $CEU_i$ such that $CEU_i \in \mathcal{R}^{u_i}(A_{-i} \mid E_{-i}^0) = \mathcal{R}^{u_i}(A_{-i})$ and $CEU_i(f^{a_i}) \geq CEU_i(g)$ for all $g \in \mathcal{F}^{A_i}$. Let $\nu_i$ be the capacity uniquely associated with $CEU_i$. Then $\nu_i \in C_i^1$. Moreover, $a_i$ is a Choquet best response w.r.t. $\nu_i$. Thus, $a_i \in R_i^1$.

Induction hypothesis: $R_i^\ell = E_i^\ell$ for all $i \in I$ and $\ell \leq k \geq 1$.

Induction step: We need to show that $R_i^{k+1} = E_i^{k+1}$ for any $i \in I$.

For any $a_i \in R_i^{k+1}$ there exists $\nu_i \in C_i^{k+1}$ for which $a_i$ is a Choquet best response. Let $\succeq_i$ be the Choquet expected utility preference relation on $\mathcal{F}^{A_i}$ w.r.t. $\nu_i$. Then $R_{-i}^k$ is $\succeq_i$-believed. This follows from $\nu_i \in C_i^{k+1}$. Moreover, $f^{a_i} \succeq_i g$ for all $g \in \mathcal{F}^{A_i}$. Let $CEU_i \in \mathcal{R}^{u_i}(A_{-i} \mid E_{-i}^k)$ such that $CEU_i$ represents $\succeq_i$ with $\nu_i$ and $u_i$. Such $CEU_i$ exists in $\mathcal{R}^{u_i}(A_{-i} \mid E_{-i}^k)$ because by the induction hypothesis $\mathcal{R}^{u_i}(A_{-i} \mid E_{-i}^k) = \mathcal{R}^{u_i}(A_{-i} \mid R_{-i}^k)$. Then $a_i \in E_i^{k+1}$.

Conversely, for any $a_i \in E_i^{k+1}$, there exist a Choquet expected utility function $CEU_i$ such that $CEU_i \in \mathcal{R}^{u_i}(A_{-i} \mid E_{-i}^k)$. By the induction hypothesis $\mathcal{R}^{u_i}(A_{-i} \mid E_{-i}^k) = \mathcal{R}^{u_i}(A_{-i} \mid R_{-i}^k)$. Let $\nu_i$ be the capacity uniquely associated with $CEU_i$. Then $\nu_i \in C_i^{k+1}$. Moreover, $a_i$ is a Choquet best response against $\nu_i$ because $CEU_i(f^{a_i}) \geq CEU_i(g)$ for all $g \in \mathcal{F}_{A_i}$. Thus, $a_i \in R_i^{k+1}$.

Since we have shown it for any finite $k \geq 1$ and we consider games with finite actions only, we also have $E_i^\infty = R_i^\infty$.

# C The Impossibility of Beliefs-Complete Structures with General Capacities

**Theorem 7** *Let $\Omega$ be a space and $\mathcal{C}(\Omega)$ be the set of capacities on $(\Omega, 2^\Omega)$. If $|\Omega| > 1$, then there exists no surjection $t : \Omega \longrightarrow \mathcal{C}(\Omega)$.*

Proof. Define for each $A \in 2^\Omega$ a capacity

$$\nu_A(B) := \begin{cases} 1 & \text{if} \quad B \in 2^\Omega \setminus \{\emptyset\} \text{ s.t. } B \supseteq A \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that indeed $\nu_A$ is a capacity for each $A \in 2^\Omega$. Denote by $\mathcal{I}(S)$ the set of such capacities.

Note that there exists a bijection $b : 2^\Omega \longrightarrow \mathcal{I}(\Omega)$ defined by $b(A) = \nu_A$ for $A \in 2^\Omega$.

Suppose to the contrary that there exists a surjection $t : \Omega \longrightarrow \mathcal{C}(\Omega)$. Since $\mathcal{I}(\Omega) \subseteq \mathcal{C}(\Omega)$, there must be also a surjection $k : \Omega \longrightarrow \mathcal{I}(\Omega)$. Since $b$ is a bijection, $b$ is invertible. Hence, the composition $b^{-1} \circ k : \Omega \longrightarrow 2^{\Omega}$ is defined and is a surjection. But this contradicts Cantor's Theorem according to which there is no surjection between the space $\Omega$ and $2^{\Omega}$ if $|\Omega| > 1$. □

# D Universal Type Space

`To do: Update Chateauneuf to Köbberling and Wakker.`

## D.1 Universal CEU-Representation Type Space

We apply universal representation type spaces of Ganguli, Heifetz, and Lee (2016) to games in strategic form with Choquet expected utility. First, we verify that Choquet expected representation type spaces satisfy the properties sufficient for the existence of the universal Choquet expected utility representation type space. The theory of Ganguli, Heifetz, and Lee (2016) applies to monotone continuous representations on measurable spaces.

**Definition 26 (Monotone continuous representation)** *A preference relation $\succeq$ on $\mathcal{F}(\Omega)$ admits a monotone continuous representation if there exists a function $V : \mathcal{F}(\Omega) \longrightarrow \mathbb{R}$ such that*

1. *Representation: For any $f, g \in \mathcal{F}(\Omega)$, $f \succeq g$ if and only if $V(f) \geq V(g)$.*
2. *Representation continuity: For any sequence of acts $\{f_n\}_{n \geq 1}$, $f_n \in \mathcal{F}(\Omega)$ for all $n$ and $f \in \mathcal{F}(\Omega)$, we have that for all $\omega \in \Omega$, $f_n(\omega) \to f(\omega)$ implies that $V(f_n) \to V(f)$.*
3. *Representation monotonicity: For $f, g \in \mathcal{F}(\Omega)$ we have that $f \geq g$ implies $V(f) \geq V(g)$.*

**Lemma 9** *Let $\succeq$ be a preference relation on $\mathcal{F}(\Omega)$ that admits a Choquet expected utility representation with a continuous capacity $\nu$. Denote it by $CEU(f) = \int_{\Omega} f d\nu$ for $f \in \mathcal{F}(\Omega)$. Then $CEU$ is a monotone continuous representation.*

Proof. A representation theorem of Choquet expected utility applicable to our setting (i.e., measurable state-spaces and measurable real-valued acts) is due to Chateauneuf (1994).[24] Moreover, it is well known that the Choquet integral satisfies representation

[24] The representation theorem does not yield a continuous capacity. An additional axiom yielding continuous capacities is necessarily technical in nature and does not add to decision theory at a conceptual level. That's why we assume continuous capacities on top of representation.

monotonicity (e.g., Marinacci and Montrucchio, 2004, Proposition 4.11). What is left to show is that it satisfies representation continuity. Representation continuity follows from the bounded convergence lemma below that is essentially a Corollary of Denneberg's (1994, Theorem 8.1) monotone convergence theorem. It makes use of continuous capacities. □

**Lemma 10 (Bounded convergence)** *Let $\nu$ be a continuous capacity on a measurable space $(\Omega, \Sigma_\Omega)$. Let $\{f_n\}$ be a sequence of nonnegative real-valued measurable bounded functions converging pointwise to $f$, for all $\omega \in \Omega$, $f_n(\omega) \to f(\omega)$. Then $\lim_{n\to\infty} \int_\Omega f_n d\nu = \int_\Omega f d\nu$.*

Proof. Define two sequences of functions $\{g_n\}$ and $\{h_n\}$ by

$$g_n(\omega) := \sup\{f_m(\omega) : m \geq n\}$$

and

$$h_n(\omega) := \inf\{f_m(\omega) : m \geq n\}.$$

Since $\{f_n\}$ is a sequence of bounded functions, each function in the sequences $\{g_n\}$ and $\{h_n\}$ are well-defined. The functions defined pointwise either by the supremum and the infimum of a sequence of measurable functions are measurable functions (e.g., Aliprantis and Border, 2007, Theorem 4.27). By construction, $\{g_n\}$ is an monotone decreasing sequence of non-negative functions and $\{h_n\}$ is a monotone increasing sequence of non-negative functions. Also by construction $\lim_{n\to\infty} h_n = f$ and $\lim_{n\to\infty} g_n = f$ and since it is well-defined $f$ and measurable (e.g., Aliprantis and Border, 2007, Theorem 4.27). Since the Choquet integral satisfies monotone representation (e.g., Marinacci and Montrucchio, 2004, Proposition 4.11) we have for each $n$,

$$\int_\Omega h_n d\nu \leq \int_\Omega f_n d\nu \leq \int_\Omega g_n d\nu.$$

Since $\nu$ is continuous, it is continuous from above and below. By the monotone convergence theorem for continuous capacities (Denneberg, 1994, Theorem 8.1),

$$\int_\Omega g_n d\nu \to \int_\Omega f d\nu$$

and

$$\int_\Omega h_n d\nu \to \int_\Omega f d\nu$$

where former follows from $\nu$ being continuous from above and latter follows from $\nu$ being continuous from below. Then it follows that $\int_\Omega f_n d\nu \to \int_\Omega f d\nu$. □

For any measurable space $\Omega$, let $\mathcal{R}(\Omega)$ be the set of Choquet expected utility representations of preferences over acts in $\mathcal{F}_\Omega$ w.r.t. to a continuous capacity. Let $\mathcal{R}(\Omega)$ be equipped with the $\sigma$-algebra generated by sets of the form

$$[f \sqsupseteq g] := \{CEU \in \mathcal{R}(\Omega) : CEU(f) \geq CEU(g)\}$$

for all $f, g \in \mathcal{F}(\Omega)$.

Let $\mathcal{R}$ be the Choquet expected utility representation class, i.e., $\mathcal{R} := \{\mathcal{R}(\Omega) : \Omega \text{ being a measurable space}\}$.

The next notion central to the construction of a universal representation type space in Ganguli, Heifetz, and Lee (2016) is image-regularity. Applied to our context of Choquet expected utility, the definition of image-regular representation class reads as follows:

**Definition 27 (Image-regular representation class)** *The Choquet expected utility representation class $\mathcal{R}$ is image-regular if for any measurable spaces $Y$ and $Z$ and any measurable function $\phi : Y \longrightarrow Z$, the map $\check{\phi} : \mathcal{R}(Y) \longrightarrow \mathcal{R}(Z)$ given by for all $CEU \in \mathcal{R}(Y)$ and all $f \in \mathcal{F}(Z)$, $\check{\phi}(CEU)(f) = CEU(f \circ \phi)$ is well-defined.*

The next steps are to verify that our context of Choquet expected utility yields indeed an image-regular representation class.

Consider any two measurable spaces $Y$ and $Z$ with a measurable function $\phi : Y \longrightarrow Z$. For any measurable real-valued act $f \in \mathcal{F}_Z$, the composite map $f \circ \phi$ is measurable (since both maps are measurable) and measurable real-valued act in $\mathcal{F}_Y$. Let $CEU_{\succeq_Y} \in \mathcal{R}(Y)$ be the Choquet expected utility representation of preference relation $\succeq_Y$ (Chateauneuf, 1994). Again, by Chateauneuf's (1994) representation theorem, we have for any $f, g \in \mathcal{F}_Z$, $CEU_{\succeq_Y}(f \circ \phi) \geq CEU_{\succeq_Y}(g \circ \phi)$ if and only if $f \circ \phi \succeq_Y g \circ \phi$.

Define $\succeq_Z$ on acts $\mathcal{F}_Z$ by for all $f, g \in \mathcal{F}_Z$ $f \succeq_Z g$ if and only if $f \circ \phi \succeq_Y g \circ \phi$. Note that $\succeq_Z$ satisfies axioms of Choquet expected utility theory because $\succeq_Y$ does. Hence by Chateauneuf (1994), there exist a Choquet expected utility representation $CEU_{\succeq_Z} \in \mathcal{R}(Z)$ that represents $\succeq_Z$ such that $CEU_{\succeq_Z}(f) \geq CEU_{\succeq_Z}(g)$ if and only if $f \succeq_Z g$.

Now let $\check{\phi} : \mathcal{R}(Y) \longrightarrow \mathcal{R}(Z)$ be defined by for all $CEU_{\succeq_Y} \in \mathcal{R}(Y)$, $\check{\phi}(CEU_{\succeq_Y}) = CEU_{\succeq_Z}$.

We observe that $\check{\phi}$ is well-defined:

**Remark 8** *For any act $f \in \mathcal{F}_Z$ and $t \in \mathbb{R}$, the set $\{z \in Z : f(z) \geq t\}$ is $\Sigma_Z$-measurable.*

Proof. This follows immediately from $f$ being a $\Sigma_Z$-measurable function. □

**Remark 9** *For any act $f \in \mathcal{F}_Z$, $t \in \mathbb{R}$, and measurable function $\phi : Y \longrightarrow Z$, the set $\{y \in Y : f(\phi(y)) \geq t\}$ is $\Sigma_Y$-measurable.*

PROOF. The composition of measurable functions is measurable. The observation is now immediate from observing that both $f$ and $\phi$ are measurable functions. $\square$

**Remark 10** *For any act $f \in \mathcal{F}_Z$, $t \in \mathbb{R}$, and measurable function $\phi : Y \longrightarrow Z$, the set $\phi^{-1}(\{z \in Z : f(z) \geq t\})$ is $\Sigma_Y$-measurable.*

PROOF. This follows immediately from $f$ being $\Sigma_Z$-measurable and $\phi$ being a measurable function. $\square$

**Remark 11** *For any act $f \in \mathcal{F}_Z$, $t \in \mathbb{R}$, and measurable function $\phi : Y \longrightarrow Z$, we have $\phi^{-1}(\{z \in Z : f(z) \geq t\}) = \{y \in Y : f(\phi(y)) \geq t\}$.*

PROOF. $y \in \phi^{-1}(\{z' \in Z : f(z') \geq t\})$ if and only if there is a $z \in Z$ such that $\phi(y) = z$ with $f(z) \geq t$ if and only if $f(\phi(y)) \geq t$ if and only if $y \in \{y' \in Y : f(\phi(y')) \geq t\}$. $\square$

For any $f \in \mathcal{F}_Z$

$$\begin{aligned} CEU_{\succeq_Z}(f) &= \int_Z f(z) d\nu_{\succeq_Z}(z) = \int_0^1 \nu_{\succeq_Z}(\{z \in Z : f(z) \geq t\}) dt \\ CEU_{\succeq_Y}(f \circ \phi) &= \int_Y f(\phi(y)) d\nu_{\succeq_Y}(y) = \int_0^1 \nu_{\succeq_Y}(\{y \in Y : f(\phi(y)) \geq t\}) dt \end{aligned}$$

where $\nu_{\succeq_Z}$ is the capacity associated with $\succeq_Z$ and $\nu_{\succeq_Y}$ is the capacity associated with $\succeq_Y$. By Chateauneuf (1994) these capacities are unique, respectively.

For any event $E \in \Sigma_Z$, set $\nu_{\succeq_Z}(E) \equiv \nu_{\succeq_Y}(\phi^{-1}(E))$. We verify:

**Lemma 11** *$\nu_{\succeq_Z}$ is a continuous capacity on $Z$.*

PROOF. Normalization: Consider first the case $E = Z$. Then $\phi^{-1}(Z) = Y$. Thus $\nu_{\succeq_Z}(Z) = \nu_{\succeq_Y}(\phi^{-1}(Z)) = \nu_{\succeq_Y}(Y) = 1$.

Second, consider $E = \emptyset$. We have $\phi^{-1}(\emptyset) = \emptyset$. (I.e., there does not exist $y \in Y$ such that $\phi(y) \in \emptyset$.) Thus $\nu_{\succeq_Z}(\emptyset) = \nu_{\succeq_Y}(\phi^{-1}(\emptyset)) = \nu_{\succeq_Y}(\emptyset) = 0$.

Monotonicity: Let $E, F \in \Sigma_Z$. If $E \subseteq F$ then $\phi^{-1}(E) \subseteq \phi^{-1}(F)$. (I.e., if $E \subseteq F$, then for all $y \in Y$ such that $\phi(y) \in E$ we also must have $\phi(y) \in F$.) By the monotonicity of $\nu_{\succeq_Y}$, $\phi^{-1}(E) \subseteq \phi^{-1}(F)$ implies $\nu_{\succeq_Y}(\phi^{-1}(E)) \leq \nu_{\succeq_Y}(\phi^{-1}(F))$. By definition of $\nu_{\succeq_Z}$ we have $\nu_{\succeq_Z}(E) \leq \nu_{\succeq_Z}(F)$.

Continuity: We need to show that for any increasing (resp. decreasing) sequence of measurable sets $\{E_n\}$, $E_n \in \Sigma_Z$ for $n = 1, 2, ...$, with $E_1 \subseteq E_2 \subseteq ...$ (resp. $E_1 \supseteq E_2 \supseteq ...$) and $\bigcup_n E_n = E$ (resp. $\bigcap_n E_n = E$), we have $\lim_{n \to \infty} \nu_{\succeq_Z}(E_n) = \nu_{\succeq_Z}(E)$.

Let $\{E_n\}$ be such that $E_n \in \Sigma_Z$ for $n = 1, 2, ...$, with $E_1 \subseteq E_2 \subseteq ...$ and $\bigcup_n E_n = E$. Then by the same arguments as used in the proof of monotonicity, we have $\phi^{-1}(E_1) \subseteq$

$\phi^{-1}(E_2) \subseteq ....$ Since $\phi$ is measurable, we have $\phi^{-1}(E_n) \in \Sigma_Y$ for $n = 1, 2, ....$ Moreover, $\bigcup_n E_n = E$ implies $\phi^{-1}(\bigcup_n E_n) = \phi^{-1}(E)$.

Claim: $\bigcup_n \phi^{-1}(E_n) = \phi^{-1}(\bigcup_n E_n)$. First, we prove "$\subseteq$". $y \in \bigcup_n \phi^{-1}(E_n)$ implies that there exist $m \in \{1, 2, ...\}$ such that $y \in \phi^{-1}(E_m)$. $E_m \subseteq \bigcup_n E_n$. By the same arguments as used in the proof of monotonicity, we have $\phi^{-1}(E_m) \subseteq \phi^{-1}(\bigcup_n E_n)$. Hence $y \in \phi^{-1}(\bigcup_n E_n)$. Next, we prove "$\supseteq$". $y \in \phi^{-1}(\bigcup_n E_n)$. There exists $z \in \bigcup_n E_n$ such that $\phi(y) = z$. There exists $m \in \{1, 2, ...\}$ such that $z \in E_m$. It follows that $y \in \phi^{-1}(E_m)$. Hence $y \in \bigcup_n \phi^{-1}(E_n)$.

We conclude $\bigcup_n \phi^{-1}(E_n) = \phi^{-1}(E)$. Thus $\lim_{n\to\infty} \nu_{\succeq_Z}(E_n) = \lim_{n\to} \nu_{\succeq_Y}(\phi^{-1}(E_n)) = \nu_{\succeq_Y}(\phi^1(E)) = \nu_{\succeq_Z}(E)$, where the equality in the middle follows from continuity of $\nu_{\succeq_Y}$.

Finally, let $\{E_n\}$ be such that $E_n \in \Sigma_Z$ for $n = 1, 2, ...$, with $E_1 \supseteq E_2 \supseteq ...$ and $\bigcap_n E_n = E$. Then by the same arguments as used in the proof of monotonicity, we have $\phi^{-1}(E_1) \supseteq \phi^{-1}(E_2) \supseteq ...$ Moreover, $\bigcap_n E_n = E$ implies $\phi^{-1}(\bigcap_n E_n) = \phi^{-1}(E)$.

Claim: $\bigcap_n \phi^{-1}(E_n) = \phi^{-1}(\bigcap_n E_n)$. Note $y \in \bigcap_n \phi^{-1}(E_n)$ if and only if $y \in \phi^{-1}(E_n)$ for all $n = 1, 2, ...$ if and only if there exists $z \in Z$ such that $\phi(y) = z$ and $z \in E_n$ for all $n = 1, 2, ...$ if and only if $z \in \bigcap_n E_n$ if and only if $y \in \phi^{-1}(\bigcap_n E_n)$.

We conclude $\bigcap_n \phi^{-1}(E_n) = \phi^{-1}(E)$. Thus $\lim_{n\to\infty} \nu_{\succeq_Z}(E_n) = \lim_{n\to} \nu_{\succeq_Y}(\phi^{-1}(E_n)) = \nu_{\succeq_Y}(\phi^1(E)) = \nu_{\succeq_Z}(E)$, where the equality in the middle follows from continuity of $\nu_{\succeq_Y}$.
□

It follows now that for any $f \in \mathcal{F}_Z$,

$$\begin{aligned}
CEU_{\succeq_Y}(f \circ \phi) &= \int_0^1 \nu_{\succeq_Y}(\{y \in Y : f(\phi(y)) \geq t\})dt \\
&= \int_0^1 \nu_{\succeq_Y}(\phi^{-1}(\{z \in Z : f(z) \geq t\}))dt \\
&= \int_0^1 \nu_{\succeq_Z}(\{z \in Z : f(z) \geq t\})dt \\
&= \check{\phi}(CEU_{\succeq_Y})(f)
\end{aligned}$$

where the second equality follows from Remark 11 and the third equality follows from the definition of $\nu_{\succeq_Z}$. We finally conclude:

**Lemma 12** *For any measurable function $\phi : Y \longrightarrow Z$, the function $\check{\phi} : \mathcal{R}(Y) \longrightarrow \mathcal{R}(Z)$ constructed above is well-defined.*

**Corollary 2** *The class of Choquet expected utility representations is image-regular.*

**Definition 28** *A Choquet expected utility representation type space is a tuple $\langle (T_i)_{i\in I}, (m_i)_{i\in I}, (s_i)_{i\in I}\rangle$ such that for each $i \in I$,*

1. *$T_i$ is a measurable space*

2. *$m_i : T_i \longrightarrow \mathcal{R}(T_{-i})$ is measurable.*

3. *$s_i : T_i \longrightarrow A_i$ is measurable*

That is, in a Choquet expected utility representation type space each type of a player is associated with a Choquet expected utility representation (with a given Bernoulli utility function) and an action.

Fix a game in strategic form $\langle I, (A_i)_{i\in I}, (u_i)_{i\in i}\rangle$. Recall that for each player $i \in I$ and each action $a_i \in A_i$, we associate the act $f^{a_i} \in \mathcal{F}(A_{-i})$ defined by

$$f^{a_i}(a_{-i}) = u_i(a_i, a_{-i}).$$

Analogously, given a Choquet expected utility representation type space $\langle (T_i)_{i\in I}, (m_i)_{i\in I}, (s_i)_{i\in I}\rangle$ we can associate to each action $a_i \in A_i$ the act $f^{a_i} \in \mathcal{F}(T_{-i})$ defined by

$$f^{a_i}(t_{-i}) = u_i(a_i, s_{-i}(t_{-i})).$$

Given a game in strategic form $\langle I, (A_i)_{i\in I}, (u_i)_{i\in i}\rangle$ and a Choquet expected utility representation type space $\langle (T_i)_{i\in I}, (m_i)_{i\in I}, (s_i)_{i\in I}\rangle$ we have for every $i \in I$, $a_i \in A_i$, and $t_i \in T_i$,

$$m_i(t_i)(u_i(a_i, s_{-i}(\cdot))) = \int_{T_{-i}} u_i(a_i, s_{-i}(t_{-i}))d\nu(t_{-i}),$$

the Choquet integral. When no confusion arises, we abuse notation and write $m_i(t_i)(a_i)$ for $m_i(t_i)(u_i(a_i, s_{-i}(\cdot)))$.

For any $a_i, a'_i \in A_i$, let

$$[a_i \sqsupseteq a'_i]^i := \{t_i \in T_i : m_i(t_i)(a_i) \geq m_i(t_i)(a'_i)\}.$$

Rewrite with slight abuse of notation our earlier definition (see page 45)

$$[a_i \sqsupseteq a'_i] = \{CEU \in \mathcal{R}(T_{-i}) : CEU(f^{a_i}) \geq CEU(f^{a'_i})\}.$$

We have

$$[a_i \sqsupseteq a'_i]^i = m_i^{-1}([a_i \sqsupseteq a'_i]).$$

**Definition 29** *Given two Choquet expected utility representation type spaces $\langle (T_i)_{i\in I}, (m_i)_{i\in I}, (s_i)_{i\in I}\rangle$ and $\langle (T'_i)_{i\in I}, (m'_i)_{i\in I}, (s'_i)_{i\in I}\rangle$, a type morphism is a function $\phi = (\phi_i)_{i\in I} : T \longrightarrow T'$ such that for each $i \in I$,*

1. *$\phi_i : T_i \longrightarrow T'_i$ is measurable,*

*2. for each $t_i \in T_i$, $m_i'(\phi_i(t_i)) = \check{\phi}_i(m_i(t_i))$, i.e., for every $a_i \in A_i$,*

$$m_i'(\phi_i(t_i))(u_i(a_i, s_{-i}'(\cdot))) = m_i(t_i)(u_i(a_i, s_{-i}'(\phi_{-i}(\cdot)))).$$

*3. $s_i' \circ \phi_i = s_i$.*

Type morphisms are interpreted as mappings that preserve the Choquet expected utility representation structure.

**Definition 30** *A Choquet expected utility representation type space $\langle (T_i^*)_{i\in I}, (m_i^*)_{i\in I}, (s_i^*)_{i\in I}\rangle$ is universal if for every Choquet expected utility representation type space $\langle (T_i)_{i\in I}, (m_i)_{i\in I}, (s_i)_{i\in I}\rangle$ there exists a unique type morphism from $\langle (T_i)_{i\in I}, (m_i)_{i\in I}, (s_i)_{i\in I}\rangle$ to $\langle (T_i^*)_{i\in I}, (m_i^*)_{i\in I}, (s_i^*)_{i\in I}\rangle$.*

In other words, the universal Choquet expected utility representation type space is the terminal object in the category of Choquet expected utility representation type spaces (that has Choquet expected utility representation type spaces as objects and type morphisms as morphisms).

**Hierarchical Construction:** For each player $i \in I$, define inductively

$H_i^0 = A_i$

and for $k \geq 1$

$H_i^{k+1} = H_i^k \times \mathcal{R}(H_{-i}^k) = H_i^0 \times \left(\times_{\ell=0}^{k} \mathcal{R}(H_{-i}^\ell)\right)$.

The space of $i$-hierarchies is

$H_i = H_i^0 \times \left(\times_{\ell=0}^{\infty} \mathcal{R}(H_{-i}^\ell)\right)$

Denote by $\rho_i^k : H_i \longrightarrow H_i^k$ the projection maps for $k \geq 0$ and $i \in I$.

Given a Choquet expected utility representation type space $\langle (T_i)_{i\in I}, (m_i)_{i\in I}, (s_i)_{i\in I}\rangle$, define for each $i \in I$ an $i$-description map $h_i : T_i \longrightarrow H_i$ as follows: $h_i^0 : T_i \longrightarrow H_i^0$ is uniquely defined by $h_i^0 = s_i$. Inductively, for $k \geq 1$ define $h_i^{k+1} : T_i \longrightarrow H_i^{k+1}$ by

$h_i^{k+1}(t_i) = \left(h_i^k(t_i), \check{h}_{-i}^k(m_i(t_i))\right) = \left(s_i(t_i), \check{h}_{-i}^0(m_i(t_i)), ..., \check{h}_{-i}^k(m_i(t_i))\right)$, where $\check{h}_i^k : \mathcal{R}(T_{-i}) \longrightarrow \mathcal{R}(H_{-i}^k)$ is the mapping between representations defined above.

Define $h_i : T_i \longrightarrow H_i$ as the unique function that satisfies for all $k \geq 0$ $h_i^k = \rho_i^k(h_i)$, i.e.,

$h_i(t_i) = \left(s_i(t_i), \check{h}_{-i}^0(m_i(t_i)), ..., \check{h}_{-i}^k(m_i(t_i)), ...\right)$.

**Proposition 7** *Type morphisms preserve i-descriptions.*

This is a variant/special case of a result in Ganguli, Heifetz, and Lee (2016). We include the proof for completeness. It specializes the proof of Ganguli, Heifetz, and

Lee (2016) to Choquet expected representation type spaces. Moreover, it modifies it in order to model uncertainty about opponents' actions rather than some abstract parameter space that in Ganguli, Heifetz, and Lee (2016) is assumed to be common to all players.

PROOF OF PROPOSITION 7. Let $\phi : T \longrightarrow T'$ be a type morphism. We have to show that $h'_i(\phi_i(t_i)) = h_i(t_i)$. We prove by induction:

Base case: For any $i \in I$, $h_i^0(t_i) = s_i(t_i) = s'_i(\phi_i(t_i)) = h'^0_i(\phi_i(t_i))$ follows directly from $\phi$ being a type morphism.

Inductive hypothesis: $h_i^k(t_i) = h'^k_i(\phi_i(t_i))$ for every $t_i \in T_i$ and $i \in I$.

Induction step: We want to show that $h_i^{k+1}(t_i) = h'^{k+1}_i(\phi_i(t_i))$ for every $t_i \in T_i$ and $i \in I$.

For any $a_i \in A_i$,

$$\begin{aligned}
\check{h}'^k_{-i}(m'_i(\phi_i(t_i)))(u_i(a_i, s'_{-i}(\cdot))) &= m'_i(\phi_i(t_i))(u_i(a_i, s'_{-i}(h'^k_{-i}(\cdot)))) \\
&= m_i(t_i)(u_i(a_i, s_{-i}(h'^k_{-i}(\phi_{-i}(\cdot))))) \\
&= m_i(t_i)(u_i(a_i, s_{-i}(h^k_{-i}(\cdot)))) = \check{h}^k_{-i}(m_i(t_i))(u_i(a_i, s_{-i}(\cdot))).
\end{aligned}$$

The second equality follows from type morphism preserving representations. The third equality is implied by the induction hypothesis and $\phi$ being a type morphism.

It follows now that

$$\begin{aligned}
h'^{k+1}_i(\phi_i(t_i)) &= \left(h'^k_i(\phi_i(t_i)), \check{h}'^k_{-i}(m'_i(\phi_i(t_i)))\right) \\
&= \left(h^k_i(t_i), \check{h}^k_{-i}(m_i(t_i))\right) = h^{k+1}_i(t_i).
\end{aligned}$$

□

Define the universal Choquet expected utility representation type space by letting $T_i^*$ to be the set of all $i$-descriptions in $H_i$, i.e., all hierarchies $t_i^* \in H_i$ for which $t_i^* = h_i(t_i)$ for some $t_i \in T_i$ in some type space $\langle (T_i)_{i\in I}, (m_i)_{i\in I}, (s_i)_{i\in I}\rangle$ for the game $\langle I, (A_i)_{i\in I}, (u_i)_{i\in I}\rangle$. Define $m_i^* : T_i^* \longrightarrow \mathcal{R}(T_{-i}^*)$ by $m_i^*(t_i^*) = \check{h}_{-i}(m_i(t_i))$, and $s_i^*(t_i^*) = s_i(t_i)$.

The following results follow now immediately from corresponding results in Ganguli, Heifetz, and Lee (2016) and the above results and construction.

**Proposition 8** $\langle (T_i^*)_{i\in I}, (m_i^*)_{i\in I}, (s_i^*)_{i\in I}\rangle$ *is a Choquet expected utility representation type space.*

**Proposition 9** *For every Choquet expected utility representation type space* $\langle (T_i)_{i\in I}, (m_i)_{i\in I}, (s_i)_{i\in I}\rangle$, *the description map* $h : T \longrightarrow T^*$ *is a type morphism.*

**Theorem 8** $\langle (T_i^*)_{i\in I}, (m_i^*)_{i\in I}, (s_i^*)_{i\in I}\rangle$ *is the universal CEU type space.*

## D.2 Universal Capacity Type Space

Our goal is to construct a universal capacity type space. The basic idea is to map CEU representation type spaces to capacity type spaces in such a way that type morphisms are preserved. Hence, the terminal object of the category of CEU type spaces would be mapped to the terminal object of the category of capacity type spaces, which would be the universal capacity type space.

We start with a preliminary observation that follows directly from arguments in the the prior section:

**Remark 12** *Any measurable function $\phi : Y \longrightarrow Z$ defines a function $\tilde{\phi} : \mathcal{C}(Y) \longrightarrow \mathcal{C}(Z)$ satisfying for any $E \in \Sigma_Z$, $\tilde{\phi}(\nu_{\succeq_Y})(E) = \nu_{\succeq_Y}(\phi^{-1}(E))$.*

**Definition 31** *Given two capacity type spaces $\langle (T_i)_{i\in I}, (\tau_i)_{i\in I}, (s_i)_{i\in I}\rangle$ and $\langle (T'_i)_{i\in I}, (\tau'_i)_{i\in I}, (s'_i)_{i\in I}\rangle$, a type morphism is a function $\phi = (\phi_i)_{i\in I} : T \longrightarrow T'$ such that for each $i \in I$,*

1. *$\phi_i : T_i \longrightarrow T'_i$ is measurable,*
2. *for each $t_i \in T_i$, $\tau'_i(\phi_i(t_i)) = \tilde{\phi}_i(t_i)$, i.e., for any $E \in \Sigma_{T'_{-i}}$, $\tau'_i(\phi_i(t_i)) = \tau_i(\phi_i^{-1}(E))$.*
3. *$s'_i \circ \phi_i = s_i$.*

For capacity type spaces, type morphisms are interpreted as mappings that preserve beliefs.

**Definition 32** *A capacity type space $\langle (T^*_i)_{i\in I}, (\tau^*_i)_{i\in I}, (s^*_i)_{i\in I}\rangle$ is universal if for every capacity type space $\langle (T_i)_{i\in I}, (\tau_i)_{i\in I}, (s_i)_{i\in I}\rangle$ there exists a unique type morphism from $\langle (T_i)_{i\in I}, (\tau_i)_{i\in I}, (s_i)_{i\in I}\rangle$ to $\langle (T^*_i)_{i\in I}, (\tau^*_i)_{i\in I}, (s^*_i)_{i\in I}\rangle$.*

In order to map structure from collections of CEU representation type spaces to collections to capacity type spaces, we start by briefly introducing the useful terminology of basic category theory (see MacLane, 1978). A category **C** consists of a collection of objects and a collection of morphisms. Each morphism $f : A \longrightarrow B$ in the category has a domain and a codomain, which are objects in the category. Composition $gf$ is defined for any two morphism $f$ and $g$ of the category such that the codomain of $f$ is the domain of $g$. Composition is associative. Finally, for each object there is an identity morphism.

Let **R** denote the collection of CEU representation type spaces and type morphisms and **C** the collection of capacity type spaces and type morphism. Consider **R** as a category as follows: The collection of objects is the collection of CEU representation type spaces. The collection of morphisms are type morphisms. Clearly, composition of type morphisms is well-defined and is associative. Moreover, for each CEU representation type space, the type morphism from the type space to itself is the identity morphism. Analogous for **C**. We observe:

**Remark 13** **R** *and* **C** *are well-defined categories.*

A terminal object of a category is an object $A$ of the category such that for each object $B$ of the category there is exactly one morphism $f : B \longrightarrow A$. The following remark is clear from the definition of universal type space:

**Remark 14** *For any category of type spaces for which objects are type spaces and morphisms are type morphisms, the terminal object is the universal type space.*

Let **C** and **D** be two categories. A functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a mapping such that for any object $A$ in category **C**, $FA$ is an object in category **D** and for any morphism $f : A \longrightarrow B$ of category **C**, $Ff : FA \longrightarrow FB$ is a morphism of category **D**. Moreover, for any two morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$, $Ffg = FfFg$. Finally, for any object $A$ of category **C**, the functor assigns to the identity morphisms $id_A$ the morphism $id_{FA}$ in the category **D**, i.e., $Fid_A = id_{FA}$.

Let $A, B$ be two objects in a category **C**. Denote by $Hom(A, B)$ the collection of morphisms with domain $A$ and codomain $B$ in the category **C**. A functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ is faithful if $F : Hom(A, B) \longrightarrow Hom(FA, FB)$ is injective. A functor $F$ is full if $F : Hom(A, B) \longrightarrow Hom(FA, FB)$ is surjective. A functor that is full and faithful it called fully faithful.

A functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ reflects a property $P$ if whenever $Ff$ has property $P$ then so has $f$. It is easy to verify that a fully faithful functor reflects the property of being a terminal object. That is, if $FA$ is a terminal object of category **D** then so is $A$ a terminal object of category **D**.

A functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ is an isomorphism if there is a functor $G$ such that $FG = id_{\mathbf{C}}$ and $GF = id_{\mathbf{D}}$ (as for instance understood as morphisms in the category of categories). Clearly, if $F$ is an isomorphism than it is fully faithful but the converse is not true.

To show the existence of a universal capacity type space the idea is to find a full and faithful functor (in fact, an isomorphism) from the category of CEU representation type spaces to the category of capacity type spaces.

**Theorem 9** *There exists a universal capacity type space, the terminal object of category* **C**.

Proof. Let **R** be the category of CEU representation type spaces and **C** be the category of capacity type spaces. Define a functor $F : \mathbf{R} \longrightarrow \mathbf{C}$ as follows:

$F\langle (T_i)_{i\in I}, (m_i)_{i\in I}, (s_i)_{i\in I}\rangle = \langle (T'_i)_{i\in I}, (\tau_i)_{i\in I}, (s'_i)_{i\in I}\rangle$ defined by for all $i \in I$

(i) $T'_i = T_i$

(ii) $s'_i = s_i$

(iii) $\tau_i(t_i)$ is the capacity associated with CEU representation $m_i(t_i)$. Since the Bernoulli utility function $u_i$ is fixed with the game $\langle I, (A_i)_{i \in I}, (u_i)_{i \in I} \rangle$ and the capacity associated with $m_i(t_i)$ is unique by Chateauneuf's (1994) representation theorem, $\tau_i(t_i)$ is well-defined. I.e., $\tau_i(t_i)$ is the capacity that satisfies for all $a_i \in A_i$, $m_i(t_i)(a_i) = \int_{T_{-i}} u_i(a_i, s_{-i}(t_{-i})) d\tau_i(t_i)(t_{-i})$.

For any $\phi : T \longrightarrow T'$ in $\mathbf{R}$, $F\phi = \phi$. I.e., the type morphism from CEU representation type space $\langle (T_i)_{i \in I}, (m_i)_{i \in I}, (s_i)_{i \in I} \rangle$ to CEU representation type space $\langle (T'_i)_{i \in I}, (m'_i)_{i \in I}, (s'_i)_{i \in I} \rangle$ is mapped by $F$ to the type morphism from capacity type space $\langle (T_i)_{i \in I}, (\tau_i)_{i \in I}, (s_i)_{i \in I} \rangle$ to capacity type space $\langle (T'_i)_{i \in I}, (\tau'_i)_{i \in I}, (s'_i)_{i \in I} \rangle$ where $\langle (T_i)_{i \in I}, (\tau_i)_{i \in I}, (s_i)_{i \in I} \rangle$ is the capacity type space that the functor $F$ assigns to the CEU representation type space $\langle (T_i)_{i \in I}, (m_i)_{i \in I}, (s_i)_{i \in I} \rangle$ and $\langle (T'_i)_{i \in I}, (\tau'_i)_{i \in I}, (s'_i)_{i \in I} \rangle$ is the capacity type space that the functor $F$ assigns to the CEU representation type space $\langle (T'_i)_{i \in I}, (m'_i)_{i \in I}, (s'_i)_{i \in I} \rangle$. That is, the functor $F$ maps the type morphism $\phi_i : T_i \longrightarrow T'_i$ to itself. Moreover, $F\check{\phi} = \tilde{\phi}$. I.e., the functor $F$ maps for every player $i \in I$, $\check{\phi}_i : \mathcal{R}(T_{-i}) \longrightarrow \mathcal{R}(T'_{-i})$ uniquely to $\tilde{\phi}_i : \mathcal{C}(T_{-i}) \longrightarrow \mathcal{C}(T'_{-i})$ (see Remark 12).

By analogous arguments, $F$ has an inverse functor $G$ such that it is an isomorphism. This follows from the one-to-one correspondence between the CEU representation and the capacity by Chateauneuf's (1994) representation theorem given that Bernoulli utility functions are fixed with the game $\langle I, (A_i)_{i \in I}, (u_i)_{i \in I} \rangle$.

Since $F$ and $G$ are isomorphisms, functor $F$ (and $G$) is fully faithful and preserves terminal objects. Hence, the universal CEU representation type space, the terminal object of the category of CEU representation type spaces, is mapped into the terminal object of the category of capacity type spaces. This is the universal capacity type space. □

It is useful to "unfold" hierarchies of beliefs for capacity type spaces analogous to hierarchies of representations for CEU representation type spaces.

For each player $i \in I$, set inductively

$L_i^0 = A_i$

and for $k \geq 1$

$L_i^{k+1} = L_i^k \times \mathcal{C}(L_{-i}^k) = L_i^0 \times \left(\times_{j=0}^k \mathcal{C}(L_{-i}^j)\right)$.

The space of $i$-hierarchies is

$L_i = L_i^0 \times \left(\times_{j=0}^\infty \mathcal{C}(L_{-i}^j)\right)$

Denote by $\pi_i^k : L_i \longrightarrow L_i^k$ the projection maps for $k \geq 0$ and $i \in I$.

Given a capacity type space $\langle (T_i)_{i \in I}, (\tau_i)_{i \in I}, (s_i)_{i \in I} \rangle$, define for each $i \in I$ an $i$-description map $\ell_i : T_i \longrightarrow L_i$ as follows: $\ell_i^0 : T_i \longrightarrow L_i^0$ is uniquely defined by $\ell_i^0 = s_i$. Inductively, for $k \geq 1$ define $\ell_i^{k+1} : T_i \longrightarrow L_i^{k+1}$ by

$\ell_i^{k+1}(t_i) = \left(\ell_i^k(t_i), \tilde{\ell}_{-i}^k(\tau_i(t_i))\right) = \left(s_i(t_i), \tilde{\ell}_{-i}^0(\tau_i(t_i)), ..., \tilde{\ell}_{-i}^k(\tau_i(t_i))\right)$, where $\tilde{\ell}_i^k : \mathcal{C}(T_{-i}) \longrightarrow$

$\mathcal{C}(L^k_{-i})$ is the mapping between capacities defined in Remark 12.

Define $\ell_i : T_i \longrightarrow L_i$ as the unique function that satisfies for all $k \geq 0$ $\ell_i^k = \pi_i^k(\ell_i)$, i.e.,

$$\ell_i(t_i) = \left( s_i(t_i), \tilde{\ell}^0_{-i}(\tau_i(t_i)), ..., \tilde{\ell}^k_{-i}(\tau_i(t_i)), ... \right).$$

**Theorem 10** *The universal capacity type space $\langle (T_i^*)_{i\in I}, (\tau_i^*)_{i\in I}, (s_i^*)_{i\in I}\rangle$ is given by for any player $i \in I$, $T_i^*$ be the set of all i-descriptions in $L_i$, i.e., all hierarchies $t_i^* \in L_i$ for which $t_i^* = \ell_i(t_i)$ for some $t_i \in T_i$ in some type space $\langle (T_i)_{i\in I}, (\tau_i)_{i\in I}, (s_i)_{i\in I}\rangle$ for the game $\langle I, (A_i)_{i\in I}, (u_i)_{i\in I}\rangle$. The type mapping $\tau_i^* : T_i^* \longrightarrow \mathcal{C}(T^*_{-i})$ is given by $\tau_i^*(t_i^*) = \tilde{\ell}_{-i}(\tau_i(t_i))$, and $s_i^*(t_i^*) = s_i(t_i)$.*

Proof. By Proposition 9 the description map $h$ in category **R** is a type morphism. Since the functor $F$ defined in the proof of Theorem 9 is an isomorphism, it preserves type morphisms and maps it to $\ell$ in **C** and the claim follows. □

# E Proofs of Section 4

## E.1 Proof of Theorem 6

Before we prove by induction, we state a construction that facilitates the proof of $R_i^k \subseteq s_i(B^k CR_i)$, $k = 1, ....$

*Construction:* Let $\langle (T_i)_{i\in I}, (\tau_i)_{i\in I}, (s_i)_{i\in I}\rangle$ be a capacity type space such that for each $i \in I$,

(i) $T_i$ is isomorphic to $A_i \times \mathcal{C}(A_{-i})$. That is, there exists a bijection $\theta_i : A_i \times \mathcal{C}(A_{-i}) \longrightarrow T_i$ with both $\theta_i$ and $\theta_i^{-1}$ measurable. We consider $A_i \times \mathcal{C}(A_{-i})$ as a measurable space with $\sigma$-algebra $2^{A_i} \otimes \Sigma_{\mathcal{C}(A_{-i})}$.

(ii) $s_i : T_i \longrightarrow A_i$ is such that for all $(a_i, \nu_i) \in A_i \times \mathcal{C}(A_{-i})$,

$$s_i(\theta_i(a_i, \nu_i)) = a_i.$$

(iii) $\tau_i : T_i \longrightarrow \mathcal{C}(T_{-i})$ is such that for all $(a_i, \nu_i) \in A_i \times \mathcal{C}(A_{-i})$,

$$\tau_i(\theta_i(a_i, \nu_i)) \left( (s_{-i})^{-1}(E) \right) = \nu_i(E) \text{ for all } E \subseteq A_{-i}.$$

Typically there might be more than one capacity type spaces $\langle (T_i)_{i\in I}, (\tau_i)_{i\in I}, (s_i)_{i\in I}\rangle$ satisfying (i) to (iii). Yet, for any such two capacity type spaces $\langle (T_i)_{i\in I}, (\tau_i)_{i\in I}, (s_i)_{i\in I}\rangle$ and $\langle (T_i')_{i\in I}, (\tau_i')_{i\in I}, (s_i')_{i\in I}\rangle$ satisfying for all $i \in I$ (i) to (iii), there exist unique type morphisms $\phi : T \longrightarrow T^*$ and $\phi' : T' \longrightarrow T^*$ to the universal capacity type space

$\langle (T^*)_{i\in I}, (\tau_i^*)_{i\in I}, (s_i^*)_{i\in I}\rangle$ (see Appendix D) that preserve $i$-description maps. In particular, it satisfies for each $i \in I$,

$$s_i^*(\phi_i(\theta_i(a_i,\nu_i))) = s_i^*(\phi_i'(\theta_i'(a_i,\nu_i))) = a_i,$$

$$\tau_i^*(\phi_i(\theta_i(a_i,\nu_i)))\left((s_{-i}^*)^{-1}(E)\right) = \tau_i^*(\phi_i'(\theta_i'(a_i,\nu_i)))\left((s_{-i}^*)^{-1}(E)\right) = \nu_i(E) \text{ for all } E \subseteq A_{-i}.$$

In fact, for each $i \in I$, $h_i^2(\phi_i(\theta_i(a_i,\nu_i))) = h_i^2(\phi_i'(\theta_i'(a_i,\nu_i))) = (a_i,\nu_i)$.

Fix a capacity type space with $\theta_i$, $i \in I$, as defined in the construction (i.e., satisfying (i) to (iii)). For each $i \in I$, define the map $\theta_i^* : A_i \times \mathcal{C}(A_{-i}) \longrightarrow T_i^*$ by $\theta_i^* = \phi_i \circ \theta_i$. For all $i \in I$, $\theta_i^*$ is measurable because both $\phi_i$ and $\theta_i$ are measurable.

We are now ready to prove the theorem:

*Proof of $R_i^1 = s_i(B^1CR_i)$:* For any $t_i \in B^1CR_i$, $s_i(t_i)$ is a Choquet best response to $\tau_i(t_i)_{A_{-i}} \in \mathcal{C}(A_{-i})$. Thus, $s_i(t_i) \in R_i^1$.

For the converse, let $a_i \in R_i^1$ by Theorem **??**. Then there exists $\nu_i \in \mathcal{C}(A_{-i})$ such that $a_i$ is a Choquet best response to $\nu_i$. Consider $t_i^* = \theta_i^*(a_i,\nu_i) \in T_i^*$. Then by construction $s_i^*(t_i^*) = a_i$ and $a_i$ is a Choquet best response to $\tau_i^*(t_i^*)_{|A_{-i}}$. Thus, $t_i^* \in B^1CR_i$.

*Induction hypothesis:* $R_i^\ell = s_i(B^\ell CR_i)$ for all $i \in I$ and $\ell = 1,...,k$.

*Induction step:* We need to show $R_i^{k+1} = s_i(B^{k+1}CR_i)$ for all $i \in I$.

By definition of $B^{k+1}CR_i$, for any $t_i^* \in B^{k+1}CR_i$,

$$\tau_i^*(t_i^*)((T_{-i}^* \setminus B^kCR_{-i}) \cup F) = \tau_i^*(t_i^*)(F)$$

for all measurable $F \subseteq B^kCR_{-i}$. Since by the induction hypothesis, $s_j^*(B^kCR_j) = R_j^k$ for all $j \in I$,

$$\tau_i^*(t_i^*)_{|A_{-i}}((A_{-i} \setminus R_{-i}^k) \cup G) = \tau_i^*(t_i^*)_{|A_{-i}}(G)$$

for all $G \subseteq R_{-i}^k$. Observe $\tau_i^*(t_i^*)_{|A_{-i}} \in C_i^{k+1}$. Moreover, $t_i^* \in B^{k+1}CR_i \subseteq B^1CR_i$ means that $s_i^*(t_i^*)$ is a Choquet best response to $\tau_i^*(t_i^*)_{|A_{-i}}$. Thus $s_i^*(t_i^*) \in R_i^{k+1}$.

For the converse, let $a_i \in R_i^{k+1}$ by Theorem **??**. There exists $\nu_i \in \mathcal{C}^{k+1}$ such that $a_i$ is a Choquet best response to $\nu_i$. We have $\nu_i \in \mathcal{C}^{k+1}$ if and only if

$$\nu_i((A_{-i} \setminus R_{-i}^k) \cup F) = \nu_i(F) \text{ for all } F \subseteq R_{-i}^k$$

if and only if

$$\tau_i^*(\theta_i^*(a_i,\nu_i))((s_{-i}^*)^{-1}((A_{-i} \setminus R_{-i}^k) \cup F)) = \tau_i^*(\theta_i^*(a_i,\nu_i))((s_{-i}^*)^{-1}(F)) \text{ for all } F \subseteq R_{-i}^k$$

if and only if, by the induction hypothesis

$$\tau_i^*(\theta_i^*(a_i,\nu_i))((T_{-i} \setminus B^kCR_{-i}) \cup E)) = \tau_i^*(\theta_i^*(a_i,\nu_i))(E) \text{ for all measurable } E \subseteq B^kCR_{-i}$$

if and only if $\theta_i^*(a_i,\nu_i) \in B^{k+1}CR_i$. By definition, $s_i^*(\theta_i^*(a_i,\nu_i)) = a_i$.

Since we have shown the equivalence for any finite number $k$ and $i \in I$, we have by Remark **??**, Theorem **??**, and the fact that $A_i$ is finite for every $i \in I$ that it also holds in the limit. Thus, for all $i \in I$, $R_i^\infty = s_i(CBCR_i)$.

### E.2 Proof of Proposition 5

`TBA, Adam.`

### E.3 Proof of Proposition 6

### E.4 Proof of Remark 6

### E.5 Proof of Conjecture 1

### E.6 Proof of Conjecture 2

### E.7 Proof of Conjecture 3

## References

[1] Ahn, D. S. (2007). Hierarchies of ambiguous beliefs, Journal of Economic Theory 136, 286–301.

[2] Aliprantis, C. and K. Border (2007). Infinite dimensional analysis, 3rd edition, Springer: Berlin.

[3] Aryal, G. and R. Stauber (2014). Trembles in extensive games with ambiguity averse players, Economic Theory 57, 1–40.

[4] Battigalli, P., Cerreia-Vioglio, S., Maccheroni, F., and M. Marinacci (2016). A note of comparative ambiguity aversion and justifiability, Econometrica 84, 1903–1916.

[5] Battigalli, P. and M. Siniscalchi (2003). Rationalization and incomplete information, B.E. Journal of Theoretical Economics (Advances) 3, Article 3.

[6] Bernheim, D. (1984). Rationalizable strategic behavior, Econometrica 52, 1007–1028.

[7] Brandenburger, A. (2003). On the existence of a "complete" possibility structure, in: Dimitry, N., Basili, M., and I. Gilboa (Eds.), Cognitive processes and economic behavior, Routledge: London, 30–34.

[8] Brandenburger, A. and E. Dekel (1993). Hierarchies of beliefs and common knowledge, Journal of Economic Theory 59, 189–198.

[9] Brandenburger, A., Friedenberg, A., and J. Keisler (2008). Admissibility in games, Econometrica 76, 307–352.

[10] Chateauneuf, A. (1994). Modeling attitudes towards uncertainty and risk through the use of Choquet integral, Annals of Operations Research 52, 3–20.

[11] Chateauneuf, A., Dana, R. A., and J.M. Tallon (2000). Risk sharing rules and equilibria with non-additive expected utilities. Journal of Mathematical Economics 61, 953 – 957.

[12] Chateauneuf, A., Eichberger, J., and S. Grant (2003). A simple axiomatization and constructive representation proof for Choquet expected utility. Economic Theory 22, 907 – 915.

[13] Chew, S.H., and E. Karni (1994). Choquet expected utility with a finite state space: Commutativity and act-independence, Journal of Economic Theory 62, 469–479.

[14] Choquet, G. (1954). Theory of capacities, Annales Institut Fourier 5, 131–295.

[15] Denneberg, D. (1994). Non-additive measure and integral, Kluver: Dordrecht.

[16] Dominiak, A. and J.-P. Lefort (2015). Agreeing to Disagree Type Results Under Ambiguity Journal of Mathematical Economics 61, 119–129.

[17] Dominiak, A. and J.-P. Lefort (2013). Agreement Theorem for Neo-Additive Beliefs, Economic Theory 52, 1–13.

[18] Di Tillo, A. (2008). Subjective expected utility in games, Theoretical Economics 3, 287–323.

[19] Dominiak, A. and J. Eichberger (2019). Games in Context: Equilibrium Under Ambiguity for Belief Functions, Mimeo.

[20] Dow, J. and S. R. Costa Werlang (1992). Uncertainty aversion, risk aversion, and the optimal choice of portfolio, Econometrica 60, 197–204.

[21] Eichberger, J. Dominiak, A. and J.-P. Leffort (2012). Agreeable trade with optimism and pessimism, Mathematical Social Sciences 64, 119 – 126.

[22] Eichberger J. and D. Kelsey (2000), Non-Additive Beliefs and Strategic Equilibria, Games and Economic Behavior 30, 183–215

[23] Eichberger J. and D. Kelsey (2014), Optimism and pessimism in games, International Economic Review 55, 483–505.

[24] Eichberger, J., Kelsey D. and B.C. Schipper (2009). Ambiguity and social interaction, Oxford Economic Papers 61, 355–379.

[25] Epstein, L. (1999). A definition of uncertainty aversion, Review of Economic Studies 66, 579–608.

[26] Epstein, L. (1997). Preference, rationalizability and equilibrium, Journal of Economic Theory 73, 1–29.

[27] Epstein, L. and T. Wang (1996). 'Beliefs about beliefs' without probabilities, Econometrica 64, 1343–1373.

[28] Ganguli, J., Heifetz, A., and B.S. Lee (2016). Universal interactive preferences, Journal of Economic Theory 162, 237–260.

[29] Ghirardato, P. and M. Le Breton (1999). Choquet rationalizability, Caltech.

[30] Ghirardato, P., Macheroni, F., Marinacci, M., and M. Siniscalchi (2003). A subjective spin on roulette wheels, Econometrica 71, 1897–1908.

[31] Ghirardato, P. and M. Marinacci (2001). Risk, ambiguity, and the separation of utility and beliefs, Mathematics of Operations Research 26, 864–890.

[32] Gilboa, I. (1987). Expected utility with purely subjective non-additive probabilities, Journal of Mathematical Economics 16, 65–88.

[33] Gilboa, I., and D. Schmeidler (1989). Maxmin expected utility with a non-unique prior, Journal of Mathematical Economics 18, 141–153.

[34] Gul, F. (1992). Savage's theorem with a finite number of states, Journal of Economic Theory 57, 99 –110.

[35] Haller, H. (2000). Non-Additive Beliefs in Solvable Games, Theory and Decision 49, 313–338.

[36] Hanany, E., Klibanoff P., and S. Mukerji (2019). Incomplete Information Games with Ambiguity Averse Players, Mimeo.

[37] Heifetz, A. (1993). The Bayesian formulation of incomplete information - the non-compact case, International Journal of Game Theory 21, 329–338.

[38] Heifetz, A., Meier, M., and B.C. Schipper (2019). Comprehensive rationalizability, University of California, Davis.

[39] Heifetz, A. and D. Samet (1998). Topology-free typology of beliefs, Journal of Economic Theory 82, 324–341.

[40] Heifetz, A. and D. Samet (1999). Coherent beliefs are not always types, Journal of Mathematical Economics 32, 475–488.

[41] Jeleva, M. (2000). Background Risk, Demand for Insurance, and Choquet Expected Utility Preferences. The Geneva Papers on Risk and Insurance Theory 25, 7 – 28.

[42] Kajii, A. and T. Ui (2006). Agreeable bets with multiple priors. Journal of Economic Theory 128, 299–305.

[43] Kelly

[44] Kelsey, D. and F. Milne (1995). The Arbitrage Pricing Theorem with Non-expected Utility Preferences, Journal of Economic Theory 65, 557 - 574.

[45] Klibanoff, P. (1996). Uncertainty, decision, and normal-form games, Northwestern University.

[46] Klibanoff, P., Marinacci, M., and S. Mukerji (2005). A smooth model of decision making under ambiguity, Econometrica 73, 1849–1892.

[47] Köbberling, V. and P. Wakker (2003). Preference Foundations for Nonexpected Utility: A Generalized and Simplified Technique. Mathematics of Operations Research 28, 395-423.

[48] Lo, K.C. (1996). Equilibrium in Beliefs under Uncertainty, Journal of Economic Theory 71, 443 - 484.

[49] Lo, K.C. (1999). Extensive Form Games with Uncertainty Averse Players. Games and Economic Behavior 28, 256 – 270.

[50] MacLane, S. (1978). Categories for the working mathematician, 2nd edition, Springer: Berlin.

[51] Marinacci, M. (2000). Ambiguous games, Games and Economic Behavior 31, 2000, 191–219.

[52] Marinacci, M. and L. Montrucchio (2004). Introduction to the mathematics of ambiguity, in: Uncertainty in Economic Theory: a collection of essays in honor of David Schmeidler's 65th birthday (I. Gilboa, ed.), 46-107, Routledge, New York, 2004.

[53] Mertens, J.F. and S. Zamir (1985). Formulation of Bayesian analysis for games with incomplete information, International Journal of Game Theory 14, 1–29.

[54] Morris, S. (1997). Alternative definitions of knowledge. In: Bacharach, M.O.L., Gerard-Varet, L.A., Mongin, P., Shin, H.S. (Eds.), Epistemic Logic and the Theory of Games and Decisions. Kluwer Academic Publishers, pp. 217–233.

[55] Mukerji, S. (1998). Ambiguity Aversion and Incompleteness of Contractual Form, The American Economic Review 5, 1207–1231.

[56] Mukerji, S. and J.-M. Tallon (2001). Ambiguity Aversion and Incompleteness of Financial Markets, Review of Economic Studies 68, 883 – 904.

[57] Mukerji, S. and J.-M. Tallon (2004). Ambiguity aversion and the absence of wage indexation, Journal of Monetary Economics 51, 653 – 670.

[58] Nehring, K. (1999), Capacities and probabilistic beliefs: a precarious coexistence, Mathematical Social Sciences 38, 197–213.

[59] Nakamura, Y. (1990). Subjective expected utility with non-additive probabilities on finite state spaces, Journal of Economic Theory 51, 346 – 366.

[60] Nishimura, K. G. and H. Ozaki (2004). Search and Knightian uncertainty, Journal of Economic Theory 119, 299 –333.

[61] Osborne, M. and A. Rubinstein (1995). A course in game theory, MIT Press.

[62] Pearce, D.G. (1984). Rationalizable strategic behavior and the problem of perfection, Econometrica 52, 1029–1050.

[63] Pinter, M. (2012). Type spaces with non-additive beliefs, Corvinus University.

[64] Riedel, F. and L. Sass (2014). Ellsberg games, Theory and Decision 76, 469–509.

[65] Samuelson, L. (1992). Dominated strategies and common knowledge, Games and Economic Behavior 4, 284–313.

[66] Sarin, R. and P. Wakker (1992). A simple axiomatization of nonadditive expected utility, Econometrica 60, 1255–1272.

[67] Schmeidler, D. (1986). Integral Representation Without Additivity, Proceedings of the American Mathematical Society 97, 255 – 261.

[68] Schmeidler, D. (1989). Subjective probability and expected utility without additivity, Econometrica 57, 571–587.

[69] Salo, A. and M. Weber (1995) Ambiguity aversion in first-price sealed-bid auctions, Journal of Risk and Uncertainty 11, 123–137.

[70] Spohn, W. (1982). How to make sense of game theory, in: Stegmüller, W., Balzer, W., and W. Spohn (eds.), Philosophy of economics, Springer-Verlag, 239–270.

[71] Tan, C.C.T. and S.R.d.C. Werlang (1988). The Bayesian foundations of solution concepts in games, Journal of Economic Theory 45, 370–391.

[72] Wald, A. (1949). Statistical Decision Functions, Annals of Mathematical Statistics, 20, 165–205.

[73] Wakker, P. (1990). Characterizing optimism and pessimism directly through comonotonicity, Journal of Economic Theory 52, 453–463.

[74] Wakker, P. (1989). Continuous subjective expected utility with non-additive probabilities, Journal of Mathematical Economics 18, 1–27.

[75] Wakker, P. and D. Deneffe (1996). Eliciting Von Neumann-Morgenstern Utilities When Probabilities Are Distorted or Unknown, Management Science 42, 1131 – 1150.

[76] Weinstein, J. (2016). The Effect of Changes in Risk Attitude on Strategic Behavior, Econometrica 84, 1881-1902.