Games in Context: Equilibrium under Ambiguity for Belief Functions*

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Abstract

We propose a new solution concept, called Context-Dependent Equilibrium Under Ambiguity (CD-EUA), for strategic games where players' beliefs may be influenced by exogenous context-related information. Players' beliefs about the strategic behavior of their opponents are represented by *belief functions*. The notion of belief functions allows us to combine exogenous context information in the spirit of Schelling (1960) with endogenous equilibrium beliefs about the opponents' behavior in analogy to the standard Nash equilibrium. For any finite strategic game, we prove existence of a CD-EUA for any context information and any degree of confidence in it. Moreover, we show continuity of the equilibrium correspondence. Finally, we illustrate how CD-EUA can be applied to different types of context information in games by explaining some stylized facts from experimental research on co-ordination.

Keywords: Strategic games, context information, non-additive beliefs, belief functions, Choquet expected utility, equilibrium under ambiguity, optimism and pessimism.

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1 Introduction

From its beginnings, the theory of decision making under uncertainty and the theory of games have been closely related. Luce and Raiffa (1957) open their chapter on *Individual Decision Making under Uncertainty* in their book *Games and Decisions*, with the following view of strategic interaction:

> "*In a game the uncertainty is entirely to the unknown decisions of the other players, and, in the model, the degree of uncertainty is reduced through the assumption that each player knows the desires of the other players and the assumption that they will take whatever actions appear to gain their ends," (Luce and Raiffa, 1957, p.275).*

Assuming that players will pursue their interests, as Luce and Raiffa (1957) suggest, has been the leading hypothesis about players' behavior in game theory since von Neumann and Morgenstern (1944) seminal contribution. Hence, information about the opponents' payoffs must play an important role for predicting behavior in games. Such payoff information may, however, not suffice to remove uncertainty completely. In particular, in non-zero-sum games, or "mixed-motive games" as Schelling (1960) calls them, uncertainty about the unknown decisions of the opponents may also be reduced by "*contextual detail*" of the game:

> "[...] *there is a danger in too much abstractness: we change the character of the game when we drastically alter the amount of contextual detail* [...]. *It is often contextual detail that can guide the players to the discovery of a stable or, at least, mutually nondestructive outcome." (Schelling, 1960, p.162)*

Contrary to the recommendations of Schelling (1960) and with little success, many experimentalists have tried to elicit behavior close to the Nash equilibrium prediction by trying to remove all "contextual detail." There are numerous laboratory and field experiments, however, showing the importance of seemingly "irrelevant details" for predicting behavior in games (e.g., Camerer, 2003; Dufwenberg, Gächter, and Hennig-Schmidt, 2011; Hopfensitz and Lori, 2016). For instance, Eichberger, Kelsey, and Schipper (2008) find a highly significant impact of players' information about their opponent being either a "game theorist" or an "old lady" on behavior in various types of games.

So far, most studies of strategic ambiguity interpret ambiguity as a lack of confidence in Nash equilibrium behavior and assume that players remain agnostic about what alternative strategies their opponents may play.[1] Ambiguity attitude determines the evaluation of a player's strategies in this case according to the worst (for pessimists) or best outcome (for optimists) that could arise from alternative non-Nash equilibrium strategies.

In this paper, we suggest a different approach for modeling strategic ambiguity. Players' beliefs are represented by *belief functions*. Belief functions allow us to include information about a "context of a game" which is not contained in the formal description of the

[1] This approach has been pursued in Dow and Werlang (1994), Klibanoff (1996), Lo (1996), Marinacci (2000), Eichberger and Kelsey (2000), Bade (2011), Eichberger and Kelsey (2014), Stauber (2019).

game. That is, we will combine the knowledge of a player about "the desires of the other players," as Luce and Raiffa (1957) suggest, with knowledge about "contextual detail" for better predictions in games as Schelling (1960) advocates. These two components will shape players' beliefs and they will affect strategic behavior.

Belief functions have been suggested and extensively studied in Shafer (1976) as a formal tool to represent weights of "evidence." Belief functions are a special type of capacities which are particularly well suited to model incomplete or partial information. He illustrates the concept of belief functions by the following example:

> "*I might have conclusive evidence, for example, that I lost my wallet in one of three places, without any clue as to which one. This calls for a belief function that assigns a degree of belief of 100% to the three places as a set, but a degree of belief of 0% to each of the three individually," (Shafer, 1990, p.2).*

This belief function reflects complete ignorance due to the lack of any specific evidence. In other situations, however, we may have more precise information than complete ignorance but less precise than probabilistic information. One prominent example is the well-known 3-color experiment of Ellsberg (1961). Given the description about the urn, we have a conclusive evidence that the number of yellow or blue balls in the urn is 60, while we have no information about the number of blue or the number of yellow balls in the urn.

To allow beliefs of players to be influenced by exogenous information about the opponents' behavior, we introduce the notion of *context-dependent belief functions*. These belief functions consist of two components: an endogenous probability distribution over the opponents' strategy choices and an exogenous belief function representing context information. A mixture of these two components with respect to a parameter, measuring confidence in the available information, defines a context-dependent belief function.

We will argue that context-dependent belief functions offer a useful tool for modeling strategic ambiguity in games with "contextual detail." First of all, belief functions are easy to handle as they can be represented by a probability distribution on the set of all subsets of a state space.[2] Hence, strategies can be evaluated by calculating expected payoffs with respect to the probability distribution associated with a belief function (see Gilboa and Schmeidler, 1994). There is no need to compute Choquet integrals in general.

Secondly, context-dependent belief functions provide an analytic framework that allows one to study how additional information about players may affect equilibrium predictions. For example, suppose that Ann joins the minimum effort game of Huyck, Battalio, and Beil (1990) with its multiple Nash equilibria. Assume that she is informed by an outside source, e.g., the experimenter, that in previous plays of the game either the highest or the second highest outcome level could be achieved. Ann's information can be represented by a belief function assigning a weight of one to all strategy profiles of the other players yielding either the highest or the second highest outcome. In contrast, her opponents face complete ignorance regarding Ann's behavior. How will Ann and her opponents interact given this context information? Will the players coordinate on one

[2]For normal-form games, a state space refers to the set of all strategy combinations of the opponent.

of the highest outcomes or will they avoid uncertainty and choose their lowest, but safe contribution level?

To study these types of questions, we introduce a new solution concept called *Context-Dependent-Equilibrium under Ambiguity* (henceforth, *CD-EUA*). In an equilibrium with context-dependent beliefs, we require that beliefs and behavior are consistent with each other in analogy to the standard Nash equilibrium. That is, players believe that their opponents choose only strategies that are best replies with respect to their respective beliefs.

As an extreme case, one obtains pure Nash equilibria in situations where player are fully confident about their endogenous probabilistic beliefs. In general, when players trust their context information, both payoffs and context information will influence the outcome of the game. For some well-known games, exogenous context information about players' behavior will shape equilibrium beliefs and, thus, lead to equilibrium behavior that cannot be obtained in a Nash equilibrium.

Another extreme case concerns beliefs without context information. When players do not trust their endogenous equilibrium beliefs and suspect that the opponents may choose some other strategy combination without a clue from context, they face pure ambiguity about the opponents' behavior. Hence, depending on their optimistic or pessimistic attitudes, they may choose strategies maximizing a weighted average of the worst or the best outcome arising from their beliefs. In this special case, behavior in a CD-EUA corresponds to that of an Equilibrium Under Ambiguity (EUA) introduced by Eichberger and Kelsey (2014).

Apart from these extreme cases, however, the interrelationship between context information and equilibrium beliefs is more subtle. The equilibrium notion proposed here (CD-EUA) reflects the assumption that equilibrium beliefs should be focussed on optimal behavior of the other players. Hence, in equilibrium, endogenous beliefs should be guided by the assumption that other players will behave optimally given their payoffs and their beliefs. Beliefs based on exogenous context information, however, cannot be expected to satisfy such mutual consistency in general. Allowing context information to influence beliefs precludes therefore a straightforward application of solution concepts such as EUA.

Our equilibrium notion allows for an interplay of equilibrium beliefs and context information with any degree of confidence. In particular, we maintain the intuition that context information matters more when confidence in optimal behavior of the other players is low and its impact wanes as confidence in equilibrium consistent behavior of the other players increases. As players' confidence in their own beliefs rises, mutually consistent beliefs about the opponents' behavior will become more important than context information, approaching eventually the consistency of Nash equilibrium beliefs.

1.1 Related literature

This paper draws on three strands of literature: (*i*) there is an extensive literature on decision-making under uncertainty outside the expected utility paradigm, (*ii*) there is also a substantial literature on belief functions, and (*iii*) there is a small literature on how to model strategic ambiguity in interactive setups. Moreover, there is no literature studying how exogenous non-probabilistic information may affect strategic behavior.[3]

The first group deals mainly with axiomatic foundations for various representations of preferences under uncertainty. In this literature one finds careful discussions of how to model ambiguity and ambiguity attitudes. In the wake of von Neumann-Morgenstern's foundation of *Expected Utility* there have been axiomatizations of *Subjective Expected Utility* by Savage (1954) and Anscombe and Aumann (1963). Following experimental critiques by Ellsberg (1961) and Kahneman and Tversky (1979), other decision criteria under uncertainty have been suggested: *Choquet Expected Utility (CEU)* by Schmeidler (1989), *Maxmin Expected Utility (MEU)* by Gilboa and Schmeidler (1989), *MEU with Optimism and Pessimism (α-MEU)* by Ghirardato, Maccheroni, and Marinacci (2004), and *Smooth Ambiguity Model* by Klibanoff, Marinacci, and Mukerji (2005).[4]

Theoretical studies of belief functions date back to Dempster (1967) and Shafer (1976).[5] The ensuing literature in statistics and artificial intelligence is too large to be reviewed here. Grabisch (2016, Chapter 7) provides a good survey of this literature.

Belief functions have also been studied in the economic literature. Jaffray and Wakker (1993) and Wakker (2000) were among the first to provide a decision-theoretic rationale for decision criteria in setups where beliefs are represented by belief functions. Gilboa and Schmeidler (1994) provide a first comprehensive study of integration methods for belief functions. Jaffray (1989) axiomatizes an α-*MEU* representation for preferences over belief functions representing incomplete information. Ghirardato (2001) uses beliefs function to model "unforeseen contingencies" with optimistic optimistic and pessimistic attitudes towards ignorance. Marinacci (1999) derives various limit laws while Ghirardato (1997) proves the Fubini Theorem for belief functions. Mukerji (1997) provides an epistemic justification for CEU preferences with belief functions.

The third group of literature tries to incorporate non-expected-utility approaches into game-theoretic analysis. Assuming aversion (pure pessimism) towards ambiguity, Dow and Werlang (1994), Haller (2000), Marinacci (2000), and Eichberger and Kelsey (2000) adopt the CEU approach, whereas Klibanoff (1996), Lo (1996), Bade (2011) and Stauber (2019) apply the MEU model.[6] All of these studies give up the Nash equilibrium assumption of players forming probabilistic beliefs about their opponents' strategy choices which coincide with actual mixed strategy choices.[7]

Two strands of literature can be distinguished according to how they deal with mixed

[3]For individual decision making, Dominiak and Lefort (2020) show that the way individuals include exogenous information about probabilities into their preferences may fundamentally affect their decisions.

[4]Comprehensive surveys of various theories of decision-making under uncertainty are provided by Gilboa and Marinacci (2013); Karni, Maccheroni, and Marinacci (2014); Machina and Siniscalchi (2014).

[5]Dempster (1967) used the term "lower probability," the name "belief function" is due to Shafer (1976).

[6]A third approach introduces ambiguity in a Bayes-Nash framework with type-contingent payoffs with CEU preferences by Eichberger and Kelsey (2004) and Dominiak and Lee (2017), with MEU preferences by Azrieli and Teper (2011) and for the Smooth Model by Klibanoff, Marinacci, and Mukerji (2005).

[7]There is also growing literature dealing with the notion of (common knowledge) rationalizability for non-expected utility preferences; see Epstein (1997) for general preferences including MEU, Dominiak and Schipper (2020) for CEU preferences, and Battigalli et al (2016) for the Smooth Model.

strategies. Following Dow and Werlang (1994), one group[8] assumes that players form non-probabilistic beliefs over strategy choices of their opponents and choose pure but not mixed strategies. Hence, in this group of papers, equilibria differ with respect to the consistency conditions between equilibrium beliefs and optimally chosen pure strategies.

Another group[9] following Klibanoff (1996) and Lo (1996) assume that players deliberately choose mixed strategies, i.e., probability distributions over pure strategies. In this view, an equilibrium is a list of mixed strategies optimally chosen by players given their beliefs about mixed strategies of the opponents that satisfies some consistency conditions. In the first approach, there is a degree of arbitrariness in the choice of pure strategies which are considered possible according to not-necessarily-probabilistic beliefs. This problem is closely related to the question of an adequate support notion for capacities or sets of priors.[10] The second approach faces the difficulty of imposing consistency between non-probabilistic beliefs over mixed strategies and the optimally chosen mixed strategies without reverting to Nash equilibrium.

In this paper, we follow the first approach where players choose pure strategies which are optimal with respect to their not necessarily probabilistic beliefs. Closest to our solution concept is the notion of Equilibrium Under Ambiguity (EUA) of Eichberger and Kelsey (2014). Both equilibrium concepts allow for optimistic and pessimistic attitudes towards strategic ambiguity, and both build on a similar consistency requirement between beliefs and behavior. There are, however, substantial conceptual differences discussed in detail below (see Remark 3.1 and Appendix 10).

Regarding formal treatments of context information in games, we are aware only of the work by Battigalli and Friedenberg (2012) and Friedenberg and Meier (2017).[11] These papers follow an epistemic approach. A "context of a game" is modeled by restricting the epistemic type structure in which the game is analyzed to hierarchies of beliefs that respect the context information. Since the "context of a game" rules out certain beliefs, this may limit the ability of players to rationalize past behavior. Battigalli and Friedenberg (2012) study how such restrictions may affect forward induction reasoning. They formalize and characterize the notion of context-dependent forward induction reasoning for extensive-form games. Friedenberg and Meier (2017) address a different problem. They consider situations where an analyst may incorrectly specify the "context of a game." That is, it is transparent to the players but not to the analyst that certain hierarchies of beliefs can be ruled out. The authors show that it may substantially affect the Bayes-Nash equilibrium analysis if the player's type structure is a strict subset of the analyst's type structure.

This paper is organized as follows. In Section 2, we introduce the concept of games with context information. In Section 3, we define the notion of CD-EUA and state our main results. In Section 5, we apply CD-EUA to analyze some well-known experiments on co-ordination. Section 6 concludes this paper. Appendix 7 collects all proofs. We

[8] This group includes Marinacci (2000), Eichberger and Kelsey (2000, 2014), and this paper.

[9] This group includes among others Bade (2011), Riedel and Sass (2014) and Stauber (2019).

[10] For a detailed discussion on this topic see e.g., Ryan (2002) as well as Eichberger and Kelsey (2014).

[11] We thank an anonymous referee for alerting us to these papers.

provide also an Online-Appendix in which we gather Appendices 8 and 9 which contain a brief review of capacities and Choquet integrals for special types of capacities, plus Appendix 10 where we compare CD-EUA and EUA in more detail.

2 Games with strategic ambiguity

In this section, we introduce the formal framework and the main concepts that we will use to study interactive behavior under uncertainty with exogenous context information.

We consider a finite strategic game $\Gamma = (I, (S_i, u_i)_{i \in I})$ consisting of a finite set of players $I = \{1, 2, \ldots, n\}$ and for each player $i \in I$, a finite set of strategies S_i and a payoff function $u_i : S_1 \times \ldots \times S_n \to \mathbb{R}$. As usual, we write $S = S_1 \times \cdots \times S_n$ to denote the set of all strategy combinations. For each player i, $S_{-i} = \times_{j \neq i} S_j$ denotes the set of all strategy combinations of the opponents of i. We denote by Σ the collection of all subsets of S, called events, excluding the empty set (i.e., $\Sigma := \{E \subseteq S : E \neq \varnothing\}$).[12] Likewise, Σ_{-i} is the set of all subsets of S_{-i} without the empty set (i.e., $\Sigma_{-i} := \{E \subseteq S_{-i} : E \neq \varnothing\}$).

In this paper, for a finite set X, we denote by $\Delta(X)$ the set of all probability distributions on X. For instance, $\Delta(S_{-i})$ is the set of all probability distributions on S_{-i}. Likewise, $\Delta(\Sigma_{-i})$ is the set of all probability distributions on Σ_{-i}.

Sometimes we will explicitly distinguish between singleton and non-singleton events in Σ_{-i}. We denote by $S(\Sigma_{-i}) := \{E \in \Sigma_{-i} : |E| = 1\}$ all singleton events and by $N(\Sigma_{-i}) := \{E \in \Sigma_{-i} : |E| \geq 2\}$ all non-singleton events in Σ_{-i}. Hence, $\Sigma_{-i} = S(\Sigma_{-i}) \cup N(\Sigma_{-i})$.

2.1 Belief functions

To model beliefs about strategic behavior, we invoke the concept of belief functions. Belief functions are a special case of (normalized and monotone) set functions, called capacities. The associated concept of integration is the Choquet integral (Choquet, 1953).

Belief functions have an intuitive definition making them convenient for applications. A belief function $\phi_i^{\gamma_i} : \Sigma_{-i} \to [0, 1]$ is a capacity with an additive representation on Σ_{-i}. We call a probability distribution γ_i on Σ_{-i} a *mass distribution*. To each mass distribution $\gamma_i \in \Delta(\Sigma_{-i})$ corresponds a unique belief function on Σ_{-i}, and *vice versa*.[13] The associated *belief function* $\phi_i^{\gamma_i}$ over Σ_{-i} can be uniquely defined as follows:[14]

$$\phi_i^{\gamma_i}(E) := \sum_{A \subseteq E} \gamma_i(A) \text{ for all } E \in \Sigma_{-i}. \tag{2.1}$$

The example below illustrates some well-known special cases of mass distributions together with their associated belief functions.

[12] Notice that, in contrast to many game-theoretic papers, Σ_i is not a set of mixed strategies on S_i. We do not consider mixed strategies in this paper.

[13] A mass distribution is not a capacity since it does not satisfy monotonicity and normalisation, see Appendix B for the formal definition of a capacity.

[14] In this paper, we develop the necessary concepts directly for belief functions. For the reader interested in how belief functions are related to general capacities, we provide a short review in the Appendices 8 and 9.

Example 2.1 (Belief functions). (*i*) *No ambiguity:* A probability distribution $\pi_i \in \Delta(S_{-i})$ has a mass distribution which is concentrated on the singleton sets in S_{-i}, i.e.,

$$\gamma_i(E) = \begin{cases} \pi_i(s_{-i}) & \text{if} \quad E = \{s_{-i}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

In this case, the associated belief function is the *probability distribution* on S_{-i},

$$\phi_i^{\gamma_i}(E) = \sum_{s_{-i} \in E} \pi_i(s_{-i}) \text{ for all } E \in \Sigma_{-i}. \tag{2.3}$$

(*ii*) *Pure ambiguity*: The mass distribution which puts all weight on S_{-i}; i.e.,

$$\gamma_i(E) = \begin{cases} 1 & \text{if} \quad E = S_{-i}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.4}$$

defines the *capacity of complete ignorance* given by

$$\phi_i^{\gamma_i}(E) = \begin{cases} 1 & \text{if} \quad E = S_{-i}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.5}$$

(*iii*) *Constant ambiguity*: For a probability distribution $\pi_i \in \Delta(S_{-i})$ and a parameter $\varepsilon \in (0, 1]$, the mass distribution

$$\gamma_i(E) = \begin{cases} \varepsilon\pi_i(s_{-i}) & \text{if} \quad E = \{s_{-i}\}, \\ 1 - \varepsilon & \text{if} \quad E = S_{-i}, \\ 0 & \text{otherwise,} \end{cases} \tag{2.6}$$

defines the *simple capacity* (or ε-*contamination*)

$$\phi_i^{\gamma_i}(E) = \begin{cases} \varepsilon \sum_{s_{-i} \in E} \pi_i(s_{-i}) & \text{if} \quad E \neq S_{-i}, \\ 1 & \text{if} \quad E = S_{-i}. \end{cases} \tag{2.7}$$

Notice that the mass distribution associated with a simple capacity (2.6) can be expressed as an ε-mixture between a probability distribution $\pi_i \in \Delta(\Sigma_{-i})$ and the mass distribution of the capacity of complete ignorance γ_i (2.4).

A belief function $\phi_i^{\gamma_i}$ is a convex capacity. Hence, it has a non-empty core:

$$\text{core}(\phi_i^{\gamma_i}) := \left\{\pi_i \in \Delta(S_{-i}) : \pi_i(E) \geq \phi_i^{\gamma_i}(E) \text{ for all } E \in \Sigma_{-i}\right\} \neq \varnothing. \tag{2.8}$$

It is the set of probability distributions which are consistent with the constraints imposed on the probabilities of events by the belief function. One can interpret the core of a belief function as a set of priors that are viewed possible by a player. The following diagram illustrates the core of a *simple capacity* for the case of three strategies of the opponent.

The area shaded in green represents the core, that is, the set of probability distributions in the simplex $\Delta(S_{-i}\ )$ which are compatible with the ε-contamination. For $\varepsilon = 0$, that is for complete ambiguity, the core equals the full simplex. For $\varepsilon = 1$ there is no ambiguity and the belief function coincides with the probability distribution π_i.

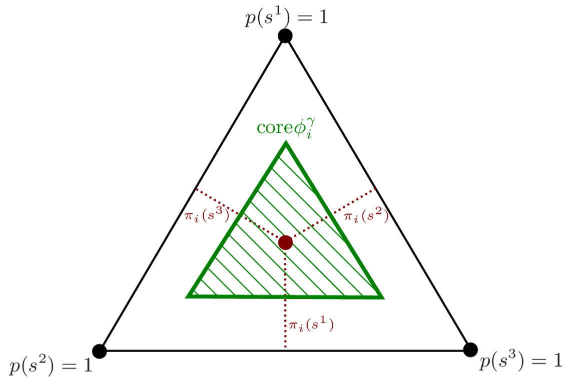

Figure 1: *Core of a simple capacity*

2.2 Choquet integral with respect to a JP-capacity

Following Schmeidler (1989), an extended literature associates *convex* capacities with aversion towards ambiguity. In this literature, the Choquet integral of a convex capacity reflects *ambiguity aversion* or *pessimism*, while the Choquet integral of a concave capacity represents an *ambiguity-seeking* or *optimistic* attitude towards ambiguity (see e.g., Schmeidler, 1989; Wakker, 1990, 2005; Ghirardato and Marinacci, 2002).

Let μ_i be a capacity and $\overline{\mu}_i$ its dual, defined by $\overline{\mu}_i(E) := 1 - \mu_i(S_{-i} \backslash E)$ for $E \subseteq S_{-i}$. For a convex capacity μ_i, the dual capacity $\overline{\mu}_i$ is concave. A capacity and its dual are characterized by the same set of compatible probability distributions. Hence, the Choquet integral of a convex capacity reflects an ambiguity-averse attitude towards the ambiguity represented by the core of the capacity, while the Choquet integral of the dual capacity yields an optimistic or ambiguity seeking evaluation of ambiguous outcomes.

Jaffray and Philippe (1997) studied capacities which are a weighted averages of a convex capacity and its concave dual capacity. Since a capacity and its dual represent the same information, a weighted average of a convex capacity and its dual has a Choquet integral which averages the optimistic and the pessimistic evaluation over the set of probability distributions in the core of the convex capacity. Consider a convex capacity μ_i and its dual capacity $\overline{\mu}_i$. We will refer to the convex combination $\alpha_i \mu_i + (1 - \alpha_i)\overline{\mu}_i$ as *J(affray)P(hilippe)-capacity*, denoted by $\nu_i^{JP}(\alpha_i, \mu_i)$. It is given by, for each $E \in \Sigma_{-i}$,

$$\nu_i^{JP}(\alpha_i, \mu_i)(E) := \alpha_i \mu_i(E) + (1 - \alpha_i)\overline{\mu}_i(E). \tag{2.9}$$

One can interpret α_i as measuring ambiguity attitude and μ_i as representing ambiguity is reflected in the multiple-prior representation of the Choquet integral with respect to a JP-capacity. More formally, the Choquet expected payoff of a strategy $s_i \in S_i$ with respect to a JP-capacity $\nu_i^{JP}(\alpha_i, \mu_i)$ over the opponents' strategies in S_{-i} is given by

$$V_i^{JP}(s_i \mid \alpha_i, \mu_i) := \int u_i(s_i, s_{-i})\, d\nu_i^{JP}(s_{-i} \mid \alpha_i, \mu_i) \tag{2.10}$$
$$= \alpha \min_{p \in \text{core}(\mu_i)} \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})\, p(s_{-i}) + (1-\alpha) \max_{p \in \text{core}(\mu_i)} \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i})\, p(s_{-i}).$$

Hence, one can view the Choquet integral of a JP-capacity as an α-MEU representation with respect to the set of priors given by the core μ_i (see Eichberger and Kelsey, 2014).

In this paper, we suggest to model players' beliefs about the opponents' strategic choice by a belief function $\phi_i^{\gamma_i}$ generated by a mass distribution γ_i. We will argue that belief functions allow one to model ambiguity in games in an intuitive and tractable way. Moreover, as we will discuss below in more detail, the concept of a mass distribution and the associated belief functions will allow us to incorporate exogenous context information into games.[15] Hence, the belief function $\phi_i^{\gamma_i}$ will represent the beliefs of a player about the strategy combination which the opponents will choose based on context information. The associated JP-capacity will reflect the players' attitudes towards ambiguity.

For the JP-capacity $\nu_i^{JP}(\alpha_i, \phi_i^{\gamma_i}) := \alpha_i \phi_i^{\gamma_i} + (1-\alpha)\overline{\phi}_i^{\gamma_i}$ based upon a belief function $\phi_i^{\gamma_i}$ and a pessimism parameter $\alpha_i \in [0,1]$, one obtains an intuitive Choquet integral.[16]

Proposition 2.1. *Let $\phi_i^{\gamma_i}$ be the belief function associated with a mass distribution γ_i. Then, for each $s_i \in S_i$:*

$$V_i^{JP}(s_i \mid \alpha_i, \phi_i^{\gamma_i}) := \int u_i(s_i, s_{-i})\, d\nu_i^{JP}(s_{-i} \mid \alpha_i, \phi_i^{\gamma_i})$$
$$= \sum_{E \subseteq S_{-i}} \gamma_i(E) \underbrace{\left[\alpha_i \min_{s_{-i} \in E} u_i(s_i, s_{-i}) + (1-\alpha_i) \max_{s_{-i} \in E} u_i(s_i, s_{-i})\right]}_{:= V_i^{\alpha}(s_i, E)}.$$

Proof. The result follows immediately from Proposition 1 in Jaffray and Philippe (1997). □

Proposition 2.1 shows that the Choquet integral with respect to the JP-capacity based on a belief function $\phi_i^{\gamma_i}$ can be expressed as the average of the α-mixtures between the max-min-payoffs $V_i^{\alpha}(s_i, E)$ weighted with the probabilities of the mass distribution γ_i.

2.3 Context-dependent beliefs

A belief function is uniquely determined by its mass distribution, i.e., a probability distribution over Σ_{-i}. If only single strategy combinations (i.e., events in $S(\Sigma_{-i})$) obtain

[15] In Section 5, we will provide examples for how context information may explain observed behavior which is inconsistent with standard Nash equilibrium analysis.

[16] The superscript JP is supposed to indicate that the Choquet integral is taken with respect to a JP-capacity, based upon a convex capacity μ, and not with respect to the capacity μ directly. More technical details can be found in the Appendices 8 and 9.

strictly positive weights then, by construction, the corresponding belief function is a probability distribution, revealing no strategic ambiguity. In general, however, the mass distribution may assign strictly positive weights to non-singleton events in $N(\Sigma_{-i})$. In this case, the associated belief function reveals ambiguity. In situations with complete ignorance, the mass distribution ascribes zero weights to all events except the full set S_{-i} which gets a weight of one, and the associate belief functions reveals pure ambiguity.

With belief functions, strategic ambiguity can be modeled by positive values of the mass distribution γ_i ascribed to non-singleton events (i.e., sets of strategy combinations). A positive weight of γ_i ascribed to a non-singleton event reflects an exogenous piece of information about the likelihood of this event combined with ignorance about the likelihood of its sub-events including each strategy combination $s_{-i} \in E$. Similarly, in the spirit of the quote from Shafer (1990) in the introduction, Gilboa and Schmeidler (1994, p.51) interpret positive weights for non-singleton events as "direct evidence" for the likelihood of these events which cannot be broken down to its subsets.

In game theory where players form beliefs about their opponents' strategy choices it is assumed that players use their knowledge about the opponents' payoffs in order to predict their behavior. Such information gleaned from payoffs is, however, in general not sufficient to determine unambiguously the behavior of the opponents. In particular, in games with multiple Nash equilibria a player cannot predict the opponents' behavior alone from knowing their payoffs and the assumption that the other players will also maximize their payoffs. Hence, to obtain better predictions, it appears sensible to consider additional information from the context of the game in the spirit of Schelling (1960).

To illustrate the main idea, consider a simplified version of the minimum-effort games studied by Huyck, Battalio, and Beil (1990) in a series of experiments.

Example 2.2 (Minimum-effort game). There are two players, $I = \{1, 2\}$. Each player has to contribute an effort level s_i from the set $S_i = \{1, 2, 3\}$ to a common cause. Payoffs are twice the minimum effort level chosen minus the individual contribution, i.e., $u_i(s_i, s_{-i}) = 2\min\{s_i, s_{-i}\} - s_{-i}$. The following payoff matrix summarizes this game.[17]

		Player 2: 1	2	3
Player 1	1	1, 1	1, 0	1, −1
	2	0, 1	2, 2	2, 1
	3	−1, 1	1, 2	3, 3

The game has three Nash equilibria in pure strategies: $\{(1,1), (2,2), (3,3)\}$. Hence, even with complete information about payoffs, assuming mutual optimality will not suffice to predict behavior in this game.[18] Equilibrium behavior is ambiguous and players must take into account additional considerations in order to coordinate their behavior. Players may be completely ignorant about their opponent's behavior putting the whole weight

[17]In Section 4, we analyze a more general version of the minimum-effort game.

[18]In fact, the experimental study of Huyck, Battalio, and Beil (1990) investigates whether players develop conventions in order to coordinate on one of these equilibria.

on $\{1, 2, 3\}$, and choose the lowest contributions $(1, 1)$ simply because this is the only strategy which guarantees each player a certain payoff of 1. On the other hand, appealing to rationality, many game theorists argue that players will coordinate on the Pareto-dominant equilibrium $(3, 3)$. Although equilibrium payoffs are strictly increasing with equilibrium contributions, players may realize that the cost of failing to coordinate is increasing with the effort level. Hence, players may expect opponents to choose an intermediate effort level and put more weight on the event $\{2, 3\}$ than on the event $\{3\}$. Belief functions allow us to take into account such considerations.[19]

To formalize the ideas of Luce and Raiffa (1957) and of Schelling (1960) that players act upon both payoff-maximizing behavior of their opponents and context information, we will assume that the mass distribution γ_i on Σ_{-i}, which defines player i's belief function $\phi_i^{\gamma_i}$, is a convex combination of two mass distributions σ_i and β_i. More precisely,

σ_i is an *endogenous* mass distribution on Σ_{-i} ascribing positive weights only to singleton events in $S(\Sigma_{-i})$, each one referring to a single strategy combination, and

β_i is an *exogenous* mass distribution on Σ_{-i} ascribing positive weights to all events including the singleton events in $S(\Sigma_{-i})$ and non-singleton events in $N(\Sigma_{-i})$.

The endogenous component σ_i will be determined in equilibrium. It captures player i's beliefs about her opponents' payoff-maximizing behavior. The exogenous component β_i is determined by context information and captures the exogenous information about the likely behavior of the opponents. These two components together with a parameter $\delta_i \in [0, 1]$ measuring the player i's *degree of confidence* in context information β_i (respectively, $(1 - \delta_i)$ her confidence in endogenous beliefs σ_i) define a *context-dependent mass distribution* $\gamma_i(\sigma_i, \beta_i, \delta_i)$ and the corresponding *context-dependent belief function* $\phi_i^{\gamma_i(\sigma_i,\beta_i,\delta_i)}$ as follows.

Definition 2.1 (Context-dependent belief functions). A player i's exogenous context information β_i, her endogenous beliefs about the opponents' strategies σ_i and her degree of confidence $\delta_i \in [0, 1]$ define the *context-dependent mass distribution* $\gamma_i(\sigma_i, \beta_i, \delta_i)$:

$$\gamma_i(E \mid \sigma_i, \beta_i, \delta_i) := (1 - \delta_i)\sigma_i(E) + \delta_i\beta_i(E) \quad \text{for each} \quad E \in \Sigma_{-i}, \tag{2.11}$$

which in turn defines a *context-dependent belief function* $\phi_i^{\gamma_i(\sigma_i,\beta_i,\delta_i)}$ on Σ_{-i} via (2.1).

When $\delta_i = 0$, player i is not confident at all about the context-related information β_i but is fully confident that her own beliefs σ_i correctly reflect the probabilities of the opponents' strategy choices. On the other hand, when $\delta_i = 1$, player i has full confidence in the available information β_i but does not trust at all her own beliefs σ_i.

[19] The experiments of Weber (2006), which we will discuss in Section 5, suggest that players coordinate beliefs according to information about previous play.

A context-dependent belief function $\phi_i^{\gamma_i(\sigma_i,\beta_i,\delta_i)}$ and the player i's degree of pessimism α_i determines a JP-capacity for which the CEU payoff of strategy $s_i \in S_i$ is given by

$$
\begin{aligned}
V_i\big(s_i \mid \alpha_i, \gamma_i(\sigma_i,\beta_i,\delta_i)\big) &= \sum_{E\subseteq S_{-i}} \gamma_i(E \mid \sigma_i,\beta_i,\delta_i) V_i^{\alpha_i}(s_i,E) \qquad (2.12)\\
&= (1-\delta_i)\sum_{s_{-i}\in S_{-i}} u_i(s_i,s_{-i})\sigma_i(\{s_{-i}\}) + \delta_i \sum_{E\subseteq S_{-i}} \beta_i(E) V_i^{\alpha_i}(s_i,E)
\end{aligned}
$$

where

$$
V_i^{\alpha_i}(s_i,E) := \left[\alpha_i \min_{s_{-i}\in E} u_i(s_i,s_{-i}) + (1-\alpha_i)\max_{s_{-i}\in E} u_i(s_i,s_{-i})\right] \qquad (2.13)
$$

(see Proposition 2.1).[20]

A list of mass distributions $\beta = (\beta_1,\ldots,\beta_n)$ represents context information for Γ. We denote by Γ_β a strategic game Γ with exogenous context information β.

Context information is considered to be an exogenous piece of information about strategic behavior that is not captured in the formal description of the game Γ. It is a characteristic of the environment in which the particular game is played. Specifying the mass distribution β_i allows for a wide range of different types of context information. A mass distribution that is concentrated on singleton events reflects *probabilistic context information*. For example, a mass distribution β_i such that $\beta_i(\{s_{-i}\}) = 1$ and $\beta_i(E) = 0$ for all other events $E \subseteq S_{-i}$ reflects probabilistic context information about a convention that the player i's opponents follow a particular strategy s_{-i}.[21]

A mass distribution assigning positive weights to non-singleton events models *non-probabilistic context information*. As an extreme case, choosing β_i such that $\beta_i(S_{-i}) = 1$ and $\beta_i(E) = 0$ for all other $E \subset S_{-i}$ reflects context information in situations of complete ignorance, i.e., without any specific information about the opponents' strategic behavior. Many other context features can be represented by the mass function β_i. In section 5, we will discuss examples of context information for well-known games from the literature.

In some situations, context information may be summarized by a joint mass distribution $\bar{\beta}$ on Σ, depicting (common) context information about joint behavior of all players. In this case, player i's context information is the marginal distribution $\beta_i = \bar{\beta}_i(\cdot|\Sigma_i)$.

3 Context-dependent equilibrium under ambiguity

In this section, we introduce a new solution concept for games with context information, called *Context-Dependent Equilibrium under Ambiguity* (henceforth, CD-EUA).

For a strategic game Γ_β, the context-related mass function β_i and the degree of confidence δ_i of player i are exogenous features of the game, whereas the strategic component of the context-dependent belief function $\phi_i^{\gamma_i(\sigma_i,\beta_i,\delta_i)}$ reflected in the mass distribution σ_i will be determined in equilibrium endogenously. To determine σ_i, we require equilibrium

[20] For notational convenience, we write $V_i(\cdot|\alpha_i, \gamma_i(\sigma_i,\beta_i,\delta_i))$ as a function of the mass distribution $\gamma_i(\sigma_i,\beta_i,\delta_i)$ rather than as a function of the associated belief function $\phi_i^{\gamma_i(\sigma_i,\beta_i,\delta_i)}$, $V_i^{JP}(\cdot|\alpha_i, \phi_i^{\gamma_i(\sigma_i,\beta_i,\delta_i)})$.

[21] Here, δ_i could be interpreted as the degree of confidence in other players sticking to this convention.

beliefs and behavior to be consistent with each other in analogy to the Nash equilibrium notion. That is, players should expect their opponents to play optimally in respect to their payoffs and beliefs. In technical terms, the equilibrium beliefs of each player should give strictly positive weights only to strategies of her opponents that are best replies.

3.1 Best replies and equilibrium

For a player i, her context information β_i, degree of confidence δ_i, degree of pessimism α_i, and her beliefs about the opponents' strategic behavior σ_i yield a context-dependent mass distribution $\gamma_i(\sigma_i, \beta_i, \delta_i)$ and the CEU payoff $V_i\big(s_i \mid \alpha_i, \gamma_i(\sigma_i, \beta_i, \delta_i)\big)$. Strategies that maximize $V_i\big(s_i \mid \alpha_i, \gamma_i(\sigma_i, \beta_i, \delta_i)\big)$ are best replies for player i.

Definition 3.1 (Best-reply correspondence). For a context-dependent mass distribution $\gamma_i(\sigma_i, \beta_i, \delta_i) \in \Delta(\Sigma_{-i})$, the best-reply correspondence of player i is given by

$$R_i(\sigma_i, \beta_i, \delta_i) := \arg\max_{s_i \in S_i} V_i\big(s_i \mid \alpha_i, \gamma_i(\sigma_i, \beta_i, \delta_i)\big). \tag{3.1}$$

Notice that $R_i(\sigma_i, \beta_i, \delta_i)$ is well-defined since S_i is finite.[22]

To define CD-EUA, we need additional notation. For a strategy combination $s_{-i} \in S_{-i}$, we denote by $\langle s_{-i} \rangle$ the set of strategies that constitute the strategy profile s_{-i}.[23]

The concept of CD-EUA is an equilibrium in (context-dependent) beliefs. More precisely, a CD-EUA is a list of context-dependent belief functions

$$\phi := \Big(\phi_1^{\gamma_1(\widehat{\sigma}_1, \beta_1, \delta_1)}, \ldots, \phi_n^{\gamma_n(\widehat{\sigma}_n, \beta_n, \delta_n)}\Big) \tag{3.2}$$

where the endogenous mass distributions $(\widehat{\sigma}_1, \ldots, \widehat{\sigma}_n)$ are determined according to the principle that for each player i, $\widehat{\sigma}_i$ assigns strictly positive weights only to strategy combinations of the opponents that are best replies with respect to their respective beliefs. That is, for each player $i \in I$,

$$\widehat{\sigma}_i(\{s_{-i}\}) > 0 \ \text{ if and only if } \ s_j \in R_j(\widehat{\sigma}_j, \beta_j, \delta_j) \ \text{ for all } \ s_j \in \langle s_{-i} \rangle . \tag{3.3}$$

We can express this equilibrium requirement in a more compact way by

$$\operatorname{supp}(\widehat{\sigma}_i) \subseteq \mathop{\times}_{j \neq i} R_j(\widehat{\sigma}_j, \beta_j, \delta_j) \tag{3.4}$$

where $\operatorname{supp}(\widehat{\sigma}_i) := \big\{s_{-i} \in S_{-i} : \sigma_i(\{s_{-i}\}) > 0\big\}$ is the support of the mass function $\widehat{\sigma}_i$.

Definition 3.2 (Context-Dependent Equilibrium Under Ambiguity). An n-tuple of context-dependent belief functions $\big(\phi_1^{\gamma_1(\widehat{\sigma}_1, \beta_1, \delta_1)}, \ldots, \phi_n^{\gamma_n(\widehat{\sigma}_n, \beta_n, \delta_n)}\big)$ constitutes a *Context-Dependent Equilibrium Under Ambiguity* for a game Γ_β if for all players $i \in I$,

$$\operatorname{supp}(\widehat{\sigma}_i) \subseteq \mathop{\times}_{j \neq i} R_j(\widehat{\sigma}_j, \beta_j, \delta_j). \tag{3.5}$$

[22] Again, for notational convenience, we write the best-reply correspondence as depending on the components $\sigma_i, \beta_i, \delta_i$ that determine the context-dependent mass distribution $\gamma_i(\sigma_i, \beta_i, \delta_i)$.

[23] That is, $\langle s_{-i} \rangle := \{s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n | (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) = s_{-i}\}$.

Notice that since $\widehat{\sigma}_i$ is a probability distribution, its support is always non-empty.

When players are fully confident in their own beliefs (i.e, $\delta_i = 0$ for all $i \in I$) and they believe that their opponents *act independently* according to a probability distribution $\widehat{\tau}_j \in \Delta(S_j)$ yielding equilibrium beliefs $\widehat{\sigma}_i(\{s_{-i}\}) := \prod_{s_j \in \langle s_{-i} \rangle} \tau_j(s_j)$ for each $s_{-i} \in S_{-i}$ which are *independent* and *mutually agreeing* with the equilibrium beliefs of all other players, then Definition 3.2 formulates a Nash equilibrium in beliefs.

Remark 3.1. It should be noted that CD-EUA is conceptually different and not a special case of the Equilibrium under Ambiguity (EUA) of Eichberger and Kelsey (2014). Although both solution concepts rest on a similar consistency requirement between equilibrium beliefs and equilibrium behavior, this requirement is implemented differently.

In a nutshell, an EUA as defined in Eichberger and Kelsey (2014) is a list of *convex capacities* $\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_n)$ for which the respective supports contain strategy combinations which are optimal with respect to the JP-capacities based on the convex capacities.

Hence, there are crucial differences between the two solution concepts. First, the endogenous component of EUA, i.e., equilibrium beliefs, is a convex capacity, whereas the endogenous part of CD-EUA is a belief function restricted by its exogenous part β_j.[24] More precisely, the endogenous part in the CD-EUA is the probability distribution $\widehat{\sigma}_j$ associated with the context-dependent belief function $\phi_j^{\gamma_j(\widehat{\sigma}_j, \beta_j, \delta_j)}$ (see Definition 2.1).

The second difference concerns the notion of support for equilibrium beliefs. The adequate notion of support for capacities is essential for modeling strategic ambiguity. Several definitions for the support notion of a capacity have been suggested in the literature (e.g., see Dow and Werlang, 1994; Marinacci, 2000; Ryan, 2002).[25] For a convex capacity μ_i, the intersection of the supports of all probability distributions in the core of μ_i provides a natural definition of the support of μ. The EUA rests on this support notion together with the non-emptiness requirement. In CD-EUA, the endogenous component is a probability distribution $\widehat{\sigma}_j$. Hence, we can apply the standard notion of the support for a probability distribution with all its advantages such as the non-emptiness of the support for any $\widehat{\sigma}_j$.

Summing up, both solution concepts use different types of beliefs and thus different support notions. For this reason, the requirement that the "expected" behavior should be optimal in equilibrium may have different consequences for either equilibrium notion. In Appendix 10, we will discuss in more detail the relation between CD-EUA and EUA.

3.2 Examples

The following examples of two-player games illustrate the potential of CD-EUA for obtaining results which differ from those obtained without context information.

Example 3.1 shows that ambiguity can be modeled as a lack of context information. A CD-EUA can describe sensible behavior other than the standard Nash equilibrium.

[24]Notice that each belief function is a convex capacity but not vice versa.

[25]Eichberger and Kelsey (2014, Appendix A) provide a detailed comparison of these support notions.

Example 3.1 (**Co-ordination game**)**.** Consider the following asymmetric co-ordination game with two players, $I = \{1,2\}$, whose strategy sets $S_1 = \{u,d\}$ and $S_2 = \{l,r\}$:

		Player 2	
		l	r
Player 1	u	$1,2$	$1,1$
	d	$0,0$	$2,1$

(3.6)

In this game players prefer to coordinate, but on different Nash equilibria. Player 1 would prefer to co-ordinate on (u,l) and Player 2 on (d,r). If there is no hint from previous interactions and there is no specific information about their behavior, then this situation may be reflected by complete-ignorance mass distributions: $\beta_1(\{l,r\}) = 1$, $\beta_1(\{l\}) = \beta_1(\{r\}) = 0$ for Player 1, and $\beta_2(\{u,d\}) = 1$, $\beta_2(\{u\}) = \beta_2(\{d\}) = 0$ for Player 2.

Denoting by $\sigma_1(\{l\}) = q$ and $\sigma_1(\{r\}) = 1-q$ Player 1's beliefs about Player 2 choosing l and r, respectively, one can derive the best-reply correspondence of Player 1 from

$$V_1(u \mid \alpha_1, \gamma_1(\sigma_1, \beta_1, \delta_1)) - V_1(d \mid \alpha_1, \gamma_1(\sigma_1, \beta_1, \delta_1)) \gtreqless 0. \tag{3.7}$$

Similarly, setting $\sigma_2(u) = p$ and $(\sigma_2(d)) = 1-p)$, for Player 2's beliefs about Player 1 choosing u, one obtains the best-reply correspondence of Player 2 from

$$V_2(r \mid \alpha_2, \gamma_2(\sigma_2, \beta_2, \delta_2)) - V_2(l \mid \alpha_2, \gamma_2(\sigma_2, \beta_2, \delta_2)) \gtreqless 0. \tag{3.8}$$

Setting $R_1(q) := R_1(q, \delta_1, \beta_1)$ and $R_2(p) := R_2(p, \delta_2, \beta_2)$, the best-reply correspondences are

$$R_1(q) = \begin{cases} \{u\} & \text{for} q > \overline{q}(\alpha_1, \delta_1) \\ \{u,d\} & \text{for } q = \overline{q}(\alpha_1, \delta_1) \\ \{d\} & \text{for} q < \overline{q}(\alpha_1, \delta_1) \end{cases} \qquad R_2(p) = \begin{cases} \{l\} & \text{for} p > \overline{p}(\alpha_2, \delta_2) \\ \{l,r\} & \text{for } p = \overline{p}(\alpha_2, \delta_2) \\ \{r\} & \text{for} p < \overline{p}(\alpha_2, \delta_2) \end{cases}$$

$$\text{with } \overline{q}(\alpha_1, \delta_1) := \tfrac{1}{2}\left(\tfrac{1-2\alpha_1\delta_1}{1-\delta_1}\right). \qquad \text{with } \overline{p}(\alpha_2, \delta_2) := \tfrac{1}{2}\left(\tfrac{1-2(1-\alpha_2)\delta_2}{1-\delta_2}\right).$$

The best-reply correspondences are illustrated in Figure 2 for $\overline{p}(\alpha_2, \delta_2) \in (0,1)$ and for $\overline{q}(\alpha_1, \delta_1) \in (0,1)$ on the left and for $\overline{p}(\alpha_2, \delta_2) > 1$ and for $\overline{q}(\alpha_1, \delta_1) < 0$ on the right.

Under full confidence in players' own beliefs and thus no strategic ambiguity (i.e., $\delta_1 = \delta_2 = 0$), one has $\overline{q}(\alpha_1, 0) = \overline{p}(\alpha_2, 0) = \frac{1}{2}$ and there are three CD-EUA: $\big(\widehat{\sigma}_1(\{l\}) = 1,\ \widehat{\sigma}_2(\{u\}) = 1\big)$, $\big(\widehat{\sigma}_1(\{l\}) = \widehat{\sigma}_1(\{r\}) = \frac{1}{2},\ \widehat{\sigma}_2(\{u\}) = \widehat{\sigma}_2(\{d\}) = \frac{1}{2}\big)$ and $\big(\widehat{\sigma}_1(\{r\}) = 1,\ \widehat{\sigma}_2(\{d\}) = 1\big)$ corresponding to the three Nash equilibria $\{(1,1), (\frac{1}{2}, \frac{1}{2}), (0,0)\}$.

However, if both players are sufficiently pessimistic and their degrees of confidence in complete ignorance are sufficiently high so that $\alpha_i\delta_i > \frac{1}{2}$ for each $i \in \{1,2\}$, then there is a unique CD-EUA $\big(\widehat{\sigma}_1(\{r\}) = 1, \widehat{\sigma}_2(\{u\}) = 1\big)$ which is not a Nash equilibrium. Due to high ambiguity, that is high confidence in pure strategic ambiguity, and strong aversion towards ambiguity both players prefer their safe strategies u and r.

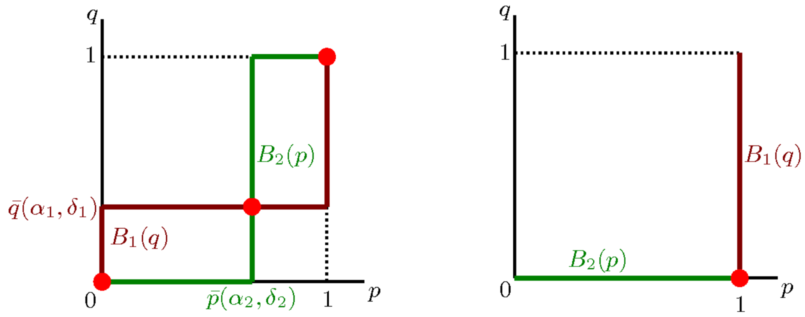

Figure 2: Best replies for $\alpha_i\delta_i < \frac{1}{2}$ (left) and $\alpha_i\delta_i > \frac{1}{2}$ (right)

Example 3.1 shows that, for sufficient ambiguity, ambiguity averse players will choose their safe strategies which appears quite sensible but cannot be modeled as a Nash equilibrium. This example demonstrates also that a lack of specific context information and complete ignorance about Nash-equilibrium behavior may obstruct successful co-ordination. In contrast, successful co-ordination on a particular Nash equilibrium may result from a convention: "play always red strategies", as the following example shows.

Example 3.2 (**Pure co-ordination game**)**.** Consider the following pure co-ordination game without conflicting interests. Two players would like to co-ordinate their behavior but do not prefer one equilibrium over the other.

		Player 2	
		L	R
Player 1	T	1,1	0,0
	B	0,0	1,1

Suppose there is a convention to "play red strategies". This probabilistic context information is depicted by the following mass distributions $\beta_1(\{L\}) = 1$, $\beta_1(\{R\}) = 0$ and $\beta_1(\{L, R\}) = 0$ for Player 1 and $\beta_2(\{T\}) = 1, \beta_2(\{B\}) = 0$ and $\beta_2(\{T, B\}) = 0$ for Player 2. Denoting by $\sigma_1(\{L\}) = q$ and by $\sigma_2(\{T\}) = p$ the beliefs of the opponents choosing L and T, respectively, and by setting $R_1(q) := R_1(q, \delta_1, \beta_1)$ and $R_2(p) := R_2(p, \delta_2, \beta_2)$, one obtains the following best-reply correspondences:

$$R_1(q) = \begin{cases} \{T\} & \text{for } q > \overline{q}(\alpha_1, \delta_1) \\ \{T, B\} & \text{for } q = \overline{q}(\alpha_1, \delta_1) \\ \{B\} & \text{for } q < \overline{q}(\alpha_1, \delta_1) \end{cases} \qquad R_2(p) = \begin{cases} \{L\} & \text{for } p > \overline{p}(\alpha_2, \delta_2) \\ \{L, R\} & \text{for } p = \overline{p}(\alpha_2, \delta_2) \\ \{R\} & \text{for } p < \overline{p}(\alpha_2, \delta_2) \end{cases}$$

with $\overline{q}(\alpha_1, \delta_1) := \frac{1}{2}\left(\frac{1-2\delta_1}{1-\delta_1}\right)$. with $\overline{p}(\alpha_2, \delta_2) := \frac{1}{2}\left(\frac{1-2\delta_2}{1-\delta_2}\right)$.

When players have low degrees of confidence in the convention $\delta_1, \delta_2 \leq \frac{1}{2}$, there are three context-dependent equilibria, two in pure strategies $(\widehat{\sigma}_1(\{L\}) = 1, \sigma_2(\{T\}) = 1)$,

$(\widehat{\sigma}_1(\{R\}) = 1, \widehat{\sigma}_2(\{B\}) = 1)$, and one "mixed" $\left(\widehat{\sigma}_1(\{L\}) = \frac{1}{2}\left(\frac{1-2\delta_1}{1-\delta_1}\right), \widehat{\sigma}_2(\{T\}) = \frac{1}{2}\left(\frac{1-2\delta_2}{1-\delta_2}\right)\right)$. These equilibria correspond to the three Nash equilibria $\{(1,1),(0,0),(\frac{1}{2},\frac{1}{2})\}$.

For high degrees of confidence in the convention to "play red strategies", $\delta_1, \delta_2 > \frac{1}{2}$, there is a unique CD-EUA $(\widehat{\sigma}_1(\{L\}) = 1, \sigma_2(\{T\}) = 1)$ where players coordinate on the "red strategies" (T, L) and thus, they behave consistent with the convention.[26]

In general, context information will not predict single strategies of the opponents but rather put constraints on strategy combinations of the other players that a player considers possible. The following example reconsiders the minimum effort game introduced in Example 2.2. It shows that context information in form of information about previous interactions may lead players to focus on a particular set of strategy combinations.

Example 2.2. (resumed) Reconsider the payoffs in the minimum-effort game:

		Player 2		
		1	2	3
Player 1	1	1, 1	1, 0	1, −1
	2	0, 1	2, 2	2, 1
	3	−1, 1	1, 2	3, 3

Without loss of generality, assume that players are pure pessimists, $\alpha_1 = \alpha_2 = 1$. Suppose that players are informed about the outcomes of such interactions in previous rounds of the game. From this information a player can deduce the minimal effort level played in the previous games, yet not the previous strategy combination. For example, assume the minimal effort level observed in previous rounds was either 2. Hence, one of the players must have chosen 2 but the other may have chosen 3. This context information can be represented by the mass distribution $\beta_i(\{2,3\}) = 1$ and $\beta_i(E) = 0$ for any other event $E \subseteq S_{-i} = \{1,2,3\}$, reflecting non-probabilistic information available to Player i.

The Choquet expected payoff of Player i from choosing an effort level $e_i \in \{1,2,3\}$ is

$$\begin{aligned} V_i(e_i \mid 1, \gamma_i(\sigma_i,\beta_i,\delta_i)) = &\sum_{E\subseteq\{1,2,3\}} \gamma_i(E|\sigma_i,\beta_i,\delta_i) \min_{e_j \in E} u_i(e_i,e_j) \\ =&(1-\delta_i)\Big[\sigma_i(\{1\})u_i(e_i,1) + \sigma_i(\{2\})u_i(e_i,2) + \sigma_i(\{3\})u_i(e_i,3)\Big] \\ &+ \delta_i \min_{e_j\in\{2,3\}} u_i(e_i,e_j) \end{aligned}$$

[26]Most social conventions are self-enforcing, that is, they prescribe behavior that is mutually optimal. It is, however, also possible to have "dysfunctional" conventions which induce inconsistent behavior. For example, the mass distributions $\beta_1(\{L\}) = 1, \beta_1(\{R\}) = 0, \beta_1(\{L,R\}) = 0$ for Player 1 and $\beta_2(\{T\}) = 1, \beta_2(\{B\}) = 0, \beta_2(\{T,B\}) = 0$ for Player 1. This probabilistic context information reflects a "dysfunctional" convention: "Player 1 plays red strategies and Player 2 black strategies". For $\delta_1 = \delta_2 > \frac{1}{2}$, it induces the unique CD-EUA $(\widehat{\sigma}_1(\{R\}) = 1, \widehat{\sigma}_2(\{T\}) = 1)$ that does not yield co-ordination.

since $\beta_i(\{2,3\}) = 1$. Hence, we have

$$V_i(e_i \mid 1, \gamma_i(\sigma_i, \beta_i, \delta_i)) = \begin{cases} 1 & \text{for} \quad e_i = 1, \\ 2((1-\delta_i)\big(\sigma_i(\{2\}) + \sigma_i(\{3\})\big) + 2\delta_i & \text{for} \quad e_i = 2, \\ (1-\delta_i)\big(\sigma_i(\{2\}) + 3\sigma_i(\{3\}) - \sigma_i(\{1\})\big) + \delta_i & \text{for} \quad e_i = 3. \end{cases}$$

It is straightforward to check that for $\delta_i > \frac{1}{2}$ and $\widehat{\sigma}_i(\{2\}) = 1$ one has

$$V_i(2 \mid 1, \gamma_i(\widehat{\sigma}_i, \beta_i, \delta_i)) = 2 > 1 = V_i(1 \mid 1, \gamma_i(\widehat{\sigma}, \beta_i, \delta_i)) = V_i(3 \mid 1, \gamma_i(\widehat{\sigma}_i, \beta_i, \delta_i)).$$

Thus,

$$\text{supp}(\widehat{\sigma}_1) = \{2\} = R_2(\widehat{\sigma}_2, \beta_2, \delta_2) \text{ and } \text{supp}(\widehat{\sigma}_2) = \{2\} = R_1(\widehat{\sigma}_1, \beta_1, \delta_1),$$

showing that $\big(\widehat{\sigma}_1(\{2\}) = 1, \widehat{\sigma}_2(\{2\}) = 1\big)$ is the unique CD-EUA.

Interestingly, experiments on behavior in the minimum-effort games conducted by Huyck, Battalio, and Beil (1990) show players choosing intermediate levels of efforts.[27]

3.3 Existence of equilibrium

In this section, we will prove existence of a CD-EUA for every finite game with n players. Moreover, we will show that the equilibrium correspondence is upper hemi-continuous with respect to the players' degrees of confidence. Hence, as all players become confident about their own beliefs, the set of CD-EUA coincides with the set of Nash equilibria.

Compared to the two-player case, the main additional problem encountered is the question of independence of beliefs about the opponents' strategies. We denote by $\tau_i \in \Delta(S_i)$ a probability distribution on the set of pure strategies S_i of player i. In games without strategic ambiguity players are assumed to choose according to a probability distributions $\tau = (\tau_1, \ldots, \tau_n) \in \times_{i \in I} \Delta(S_i)$ independently, if player i believes that a particular pure strategy combination $s_{-i} \in S_{-i}$ is played with the probability

$$\prod_{s_j \in \langle s_{-i} \rangle} \tau_j(s_j) = \tau_1(s_1) \times \cdots \times \tau_{i-1}(s_{i-1}) \times \tau_{i+1}(s_{i+1}) \times \cdots \times \tau_n(s_n). \tag{3.9}$$

where $s_j \in S_j$ is the strategy of player $j \neq i$ in the strategy combination s_{-i}. Proceeding in this way assumes not only that players believe that opponents choose strategies independently but also that all players mutually agree in their beliefs upon the probabilities with which a particular player i chooses a pure strategy from S_i.

Remark 3.2 (Correlated equilibria). These assumptions about strategic independence and of mutual agreement on beliefs stand in strong contrast to the assumption that players may feel ambiguity about the opponents' strategy choices. However, strategic

[27]In Section 5.3, we will show that such behavior can be modeled by an CD-EUA with context information.

independence is a defining property of Nash equilibrium. Relaxing this assumption would make it impossible to compare CD-EUA with Nash equilibrium when ambiguity vanishes.

For the case of multiple prior models, Lo (2009) argues convincingly that, with ambiguity, beliefs about strategies will in general be correlated. To deal with correlations among strategies in an equilibrium without ambiguity properly, one needs to model players' information about other players' payoffs by a type space with a common prior distribution (Harsanyi, 1967-68). For games with incomplete information, however, Forges (2006) distinguishes five notions of correlated equilibrium depending on whether the correlation arises from a "correlation device" or from "recommendation by a mediator". [28]

The concept of CD-EUA offers an intermediate way to deal with this problem: one can maintain independence and mutual agreement for the endogenous part of players' beliefs, yet allow for uncertainty about strategic independence reflected in the exogenous component. In this way, we can approximate the set of Nash equilibria as players' confidence in their beliefs increases.[29]

For a strategic game Γ_β with context information $\beta = (\beta_1, \ldots, \beta_n)$ and a list of parameters $\delta = (\delta_1, \ldots, \delta_n) \in [0, 1)^n$ measuring players' confidence in this information, we can determine a mass distribution σ_i for each player $i \in I$ concentrating on singletons as follows.

Definition 3.3 (Endogenous beliefs). Given a vector of probability distributions $\tau = (\tau_1, \ldots, \tau_n) \in \times_{i \in I} \Delta(S_i)$, we define the endogenous mass distribution $\sigma_i(\tau) : \Sigma_{-i} \to [0, 1]$ as a function of τ by

$$\sigma_i(\{s_{-i}\} \,|\tau) := \prod_{s_j \in \langle s_{-i} \rangle} \tau_j(s_j) \tag{3.10}$$

for each singleton event $\{s_{-i}\} \in S(\Sigma_{-i})$ and $\sigma_i(E|\tau) = 0$ for all other events $E \in N(\Sigma_{-i})$.

The endogenous component σ_i obviously satisfies independence and mutual agreement.

The mass distribution $\gamma_i(\sigma_i(\tau), \beta_i, \delta_i)$ defined as a δ_i-mixture of $\sigma_i(\tau)$ and β_i (see 2.1) and the corresponding belief function $\phi_i^{\gamma_i(\sigma_i(\tau), \beta_i, \delta_i)}$ yield the Choquet expected payoff characterized in Proposition 2.1 as a function of $\sigma_i(\tau)$ and thus, as a function of τ. That is, for each strategy $s_i \in S_i$:

$$V_i(s_i \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i, \delta_i)) := \sum_{E \subseteq S_{-i}} \gamma_i(E \mid \sigma_i(\tau), \beta_i, \delta_i) V_i^\alpha(s_i, E). \tag{3.11}$$

The Choquet expected payoff is a continuous function with respect to τ and thus we can apply standard arguments to prove the following existence result.

[28] A proper treatment of ambiguity in the context of a game of incomplete information is beyond the scope of this paper.

[29] We see it as an advantage of the suggested concept of CD-EUA that it avoids the complex, and so far unresolved, correlation problem. Finding an epistemic model which justifies a more general notion of equilibrium under ambiguity is an important problem of future research.

Theorem 3.1. *Consider a strategic game Γ_β with context information $\beta = (\beta_1, \ldots, \beta_n)$. For each vector of confidence parameters $\delta = (\delta_1, \ldots, \delta_n) \in [0,1)^n$, there exist probability distributions $\widehat{\tau} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_n) \in \times_{i \in I} \Delta(S_i)$ for which the n-tuple of belief functions*

$$\left(\phi_1^{\gamma_1(\sigma_1(\hat{\tau}), \beta_1, \delta_1)}, \ldots, \phi_n^{\gamma_n(\sigma_n(\hat{\tau}), \beta_n, \delta_n)}\right)$$

defined by the mass distributions $\left(\gamma_1(\sigma_1(\hat{\tau}), \beta_1, \delta_1), \ldots, \gamma_n(\sigma_n(\hat{\tau}), \beta_n, \delta_n)\right)$ *constitutes a* Context-Dependent Equilibrium Under Ambiguity *for* Γ_β.

Proof. See Appendix 7. □

Theorem 3.1 shows that a CD-EUA exits for any finite game with context information $\beta = (\beta_1, \ldots, \beta_n)$ and any degrees of confidence $\delta = (\delta_1, \ldots, \delta_n) \in [0,1)^n$ that players may have about the available information about their opponents' strategic behavior.

When $\delta_i = 0$ for all $i \in I$, players are fully confident about their own beliefs and hence about τ, the CEU expected payoff $V_i(s_i \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i, 0))$ of a pure strategy $s_i \in S_i$ reduces to the expected payoff with respect to $\sigma_i(\tau)$. That is,

$$V_i(s_i \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i, 0)) = \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \sigma(\{s_i\} | \tau). \tag{3.12}$$

When all players are fully confident about their endogenous beliefs (i.e., $\delta_i = 0$ for all $i \in I$) then the set of CD-EUA coincides with the set of Nash equilibria.

Proposition 3.1. *Consider a strategic game Γ_β with context information $\beta = (\beta_1, \ldots, \beta_n)$. The vector of probability distributions $\hat{\tau} = (\hat{\tau}_1, \ldots, \hat{\tau}_n) \in \times_{i \in I} \Delta(S_i)$ defines a CD-EUA*

$$(\phi_1^{\gamma_1(\sigma_1(\hat{\tau}), \beta_1, 0)}, \ldots, \phi_n^{\gamma_n(\sigma_n(\hat{\tau}), \beta_n, 0)})$$

with the confidence vector $\delta = (\delta_1, \ldots, \delta_n) = \mathbf{0}$ if and only if $\hat{\tau}$ is a Nash equilibrium.

Proof. See Appendix 7. □

Note that for $\delta = (\delta_1, \ldots, \delta_n) \neq \mathbf{0}$, the set of CD-EUA does not coincide in general with the set of Nash equilibria. Whether a CD-EUA has equilibrium beliefs corresponding to a Nash equilibrium depends on context information and players' degrees of confidence.

A useful property of CD-EUA is the upper hemi-continuity of the equilibrium correspondence with respect to the vector of confidence parameters δ.

Definition 3.4 (**Equilibrium Correspondence**). For given context information $\beta = (\beta_1, \ldots, \beta_n)$, the equilibrium correspondence $\Phi : [0,1)^n \rightrightarrows \times_{i \in I} \Delta(S_i)$ assigns to each vector of confidence parameters $\delta = (\delta_1, \ldots, \delta_n) \in [0,1]^n$ a list of probability distributions $\widehat{\tau} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_n) \in \times_{i \in I} \Delta(S_i)$ for which the belief functions constitute a CD-EUA, i.e.,

$$\Phi(\delta) := \left\{ \widehat{\tau} \in \times_{i \in I} \Delta(S_i) : \ \left(\phi_1^{\gamma_1(\sigma_1(\widehat{\tau}), \beta_1, \delta_1)}, \ldots, \phi_n^{\gamma_n(\sigma_n(\widehat{\tau}), \beta_n, \delta_n)}\right) \text{ is a CD-EUA} \right\}.$$

By Theorem 3.1, the equilibrium correspondence $\Phi(\delta)$ is not empty for any vector $\delta = (\delta_1, \dots, \delta_n) \in [0, 1)^n$ and, by Proposition 3.1, $\Phi(\mathbf{0})$ is the set of Nash equilibria.

Proposition 3.2. *The equilibrium correspondence Φ is upper hemi-continuous on $[0, 1)^n$.*

Proof. See Appendix 7. □

An immediate consequence of Theorem 3.1 is the following corollary.

Corollary 3.1. *For any game Γ_β with context information β and any sequence $\delta^\kappa = (\delta_1^\kappa, \dots, \delta_n^\kappa) \in [0, 1)^n$ converging to $\mathbf{0}$, any converging sequence of CD-EUA $\tau^\kappa \in \Phi(\delta^\kappa)$ converges to a Nash equilibrium, i.e., $\lim_{\kappa \to \infty} \tau^\kappa \in \Phi(\mathbf{0})$.*

One may wonder what happens if $\delta \to \mathbf{1}$, that is all players are fully confident about the exogenous context information. For $\delta_i = 1$, player i's beliefs adhere to her context information β_i, that is, the mass distribution $\gamma_i = \beta_i$. In this case, all players will choose a best reply given these exogenous beliefs. There will be no consistency among beliefs. For context information reflecting complete ignorance (i.e., $\beta_i(S_{-i}) = 1$ and $\beta_i(E) = 0$ for any other event E), extremely pessimistic players with $\alpha_i = 1$ will choose a maxmin strategy while extremely optimistic players with $\alpha_1 = 0$ will choose a maxmax strategy. Yet, without further assumptions about context information, no consistency can be expected.

4 Equilibrium selection

Many games are charactized by a large number of strict Nash equilibria. Since CD-EUA are Nash equilibria for $\delta = 0$ (Proposition 3.1), it appears natural to explore the potential of context information in a CD-EUA as a selection criterion in case of multiplcity of Nash equilibrium. A comprehensive study of equilibrium selection by CD-EUA is beyond the scope of this paper. Hence, in this section, we will investigate the question of equilibrium selection in the context of the minimum effort game of Huyck, Battalio, and Beil (1990), one of the prime examples for multiplicity of Nash equilibrium that has been studied extensively both theoretically and experimentally.

The selection idea is as follows. Because all pure-strategy equilibria of the minimum effort game are strict, Proposition 3.2 implies that, for small degrees of confidence in the context information, that is for δ close to 0, CD-EUAs will have the same equilibrium strategies as the nearby Nash equilibria. Context information identifying a subset E of strategy combinations[30] will select strategies that are equilibrium strategies of CD-EUAs for all degrees of confidence δ in this context information (Proposition 4.1 below). Moreover, for δ sufficiently high, only strategy combinations in E will be played in a CD-EUA and, hence, in a Nash equilibrium (Proposition 4.2 below).[31] In this sense context information selects among multiple Nash equilibria.

[30] I.e., for which $\beta_i(E) = 1$ for all $i \in I$ holds.

[31] Note that, for the minimum effort game, all strategy combinations played in a CD-EUA are Nash equilibrium strategy combinations.

Consider a minimum-effort game with n players and possible contribution levels, $T := \{0, 1, 2, 3, \ldots, 10\}$. We focus on the symmetric version of the game with strategy sets $S_i = T$ for every player $i \in I$. For $A > c > 0$, the payoff function of a player i is[32]

$$u_i(s_i, s_{-i}) = A \min \{s_i, \underline{m}(s_{-i})\} - cs_i \tag{4.1}$$

where $\underline{m}(s_{-i}) := \min\{s_j \in \langle s_{-i} \rangle : j \neq i\}$ denotes the lowest contribution of the opponents in the strategy combination s_{-i}. For every contribution level $y \in T$, the strategy combination $s^y := (y, y, \ldots, y) \in S = T^n$ where every player i chooses $s_i = y$ is a pure strategy, symmetric Nash equilibrium (see Huyck, Battalio, and Beil, 1990). Hence, there are eleven symmetric Nash equilibria collected in the set $\mathcal{N}(T) = \{s^y \in S \mid y \in T\}$. We denote by s^y_{-i} the strategy combination of the opponents where each player $j \neq i$ chooses $s_j = y \in T$.

Suppose now that players know from the context[33] of the game that contribution levels were ranging between a minimum of m and a maximum of M. Let $C := \{y \in T |\; m \leq y \leq M\}$ represent this context information for some $m, M \in T$ such that $m < M$. We denote by $\mathcal{C} = \{s^y \in S \mid y \in C\}$ the set of all strategy profiles that are consistent with C. Notice that $\mathcal{C} \subseteq \mathcal{N}(T)$.

Players $i \in I$ take context information into account via a mass distribution β_i on Σ_{-i} reflecting ambiguity with respect to the strategy combinations in $\mathcal{C}_{-i} = \{s^y_{-i} \in S_{-i} \mid y \in C\}$ which the opponents may chose. That is, $\beta_i(\mathcal{C}_{-i}) = 1$ and $\beta_i(E_{-i}) = 0$ for any other $E_{-i} \in \Sigma_{-i}$ such that $E_{-i} \neq \mathcal{C}_{-i}$.

Since players are symmetric, we will concentrate on symmetric equilibria and assume symmetric attitudes towards ambiguity and degrees of confidence, i.e., $\alpha_i = \alpha$ and $\delta_i = \delta$ for all $i \in I$. For notational convenience, we will drop the index i for these parameters.

We denote by $\mathcal{M}(C, \alpha, \delta)$ the set of pure-strategy symmetric CD-EUAs associated with context information[34] C, players' degree of confidence $\delta \in [0, 1]$ and their degree of ambiguity attitudes $\alpha \in [0, 1]$. For $s^y \in \mathcal{M}(C, \alpha, \delta)$, each player $i \in I$ believes that the opponents play the strategy combination s^y_{-i} and best replies by choosing s^y_i.

In this environment, we explore the relationship between context information C and equilibrium behavior in $\mathcal{M}(C, \alpha, \delta)$. Will context information C affect equilibrium behavior in a systematic way? We will show that context information C provides a focal set for contributions in the set $\mathcal{M}(C, \alpha, \delta)$ of CD-EUAs in the following sense.

For any context information C,

- there is a CD-EUA consistent with this context information for any degree of confidence $\delta \in [0, 1]$, i.e., $\mathcal{M}(C, \alpha, \delta) \cap \mathcal{C} \neq \emptyset$ (Proposition 4.1), and
- there is a critical level of confidence $\overline{\delta}$ such that for any $\delta \geq \overline{\delta}$, the CD-EUAs are consistent with the context information, i.e., $\mathcal{M}(C, \alpha, \delta) \subset \mathcal{C}$ (Proposition 4.2).

[32]The special case of $A = 2$, $c = 1$, and $T = \{1, 2, 3\}$ has been treated in Example 2.2 above.

[33]We abstract here from where such context information may be derived. See Section 5.3 for an explicit example.

[34]Given the assumption $\beta_i(\mathcal{C}_{-i}) = 1$, we suppress the reference to β_i in favor of the set C.

Proposition 4.1. *Consider a minimum-effort game with symmetric Nash equilibria $\mathcal{N}(T)$. Let C be context information about (equilibrium) behavior. Then, for any degree of ambiguity attitude $\alpha \in [0,1]$ and any degree of confidence $\delta \in [0,1]$, there exists a symmetric CD-EUA that is consistent with the context information C, i.e.,*

$$\mathcal{M}(C, \alpha, \delta) \cap \mathcal{C} \neq \emptyset. \tag{4.2}$$

Proof. See Appendix 7. □

Proposition 4.2. *Consider a minimum-effort game with symmetric Nash equilibria $\mathcal{N}(T)$. Let C be context information about (equilibrium) behavior. Then, for any degree of ambiguity attitude $\alpha \in [0,1]$, there is a confidence level $\overline{\delta}(\alpha) \in (0,1)$ such that for all $\delta \geq \overline{\delta}(\alpha)$, all symmetric CD-EUAs are consistent with C, i.e.,*

$$\mathcal{M}(C, \alpha, \delta) \subset \mathcal{C} \tag{4.3}$$

Proof. See Appendix 7. □

Neither Proposition 4.1 nor Proposition 4.2 do not predict precisely which CD-EUA will arise in the light of the available context information. In the spirit of Schelling (1960), however, these propositions suggest which equilibrium constellations will be focal and, hence, are likely to be observed in applications. Proposition 4.1 identifies the Nash equilibrium strategies which will be CD-EUAs regardless of the degree of confidence in this context information δ. Proposition 4.2 shows that, depending on the degree of pessimism α, there is always a critical degree of confidence $\overline{\delta}(\alpha)$ such that all CD-EUAs will be consistent with the context information C if the degree of confidence regarding the context informaion δ exceeds this critical value. Not surprisingly, this critical value for the degree of confidence will depend on the parameters of the payoff functions, $\bar{\delta}(\alpha) := \min \left\{ \frac{1}{\alpha} \left(1 - \frac{c}{A}\right), \frac{1}{(1-\alpha)} \frac{c}{A} \right\}$.

The top four diagrams in Figure 3 illustrates the best-reply correspondences of a player given the minimum contributions y of the other players for different degrees of confidence $\delta \in \left\{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}\right\}$ and given context information $C = \{4, 5, 6\}$. Blue dots represent best replies for low confidence ($\delta = \frac{1}{4}$), yellow stars indicate best replies for medium confidence ($\delta = \frac{1}{2}$), and red dots are best replies for high degrees of confidence ($\delta = \frac{3}{4}$). Equilibrium contribution of CD-EUAs lie on the diagonal (see Figure 4).

The bottom α-δ diagram of Figure 3 shows the parameter regions for the different types of best-reply correspondences and equilibrium contributions. Black dots indicate the α-δ-parameter constellations corresponding to the best replies illustrated in the top four diagrams. Notice that for low degrees of confidence $\delta \leq \frac{1}{3}$, i.e., below the green line, context information has no selective power. In contrast, for $\delta \geq \frac{2}{3}$, i.e., above the brown line, context information determines the best reply completely: for pessimism to the right of the yellow vertical line, $\alpha \geq \frac{2}{3}$, the lowest contribution level m in C will be optimal, for sufficient optimism, $\alpha < \frac{2}{3}$, the highest contribution level M will be optimal. Notice also that s^m and s^M are equilibrium profiles for pessimists and optimists, respectively, regardless of their degrees of confidence. This suggests that a CD-EAU selects among

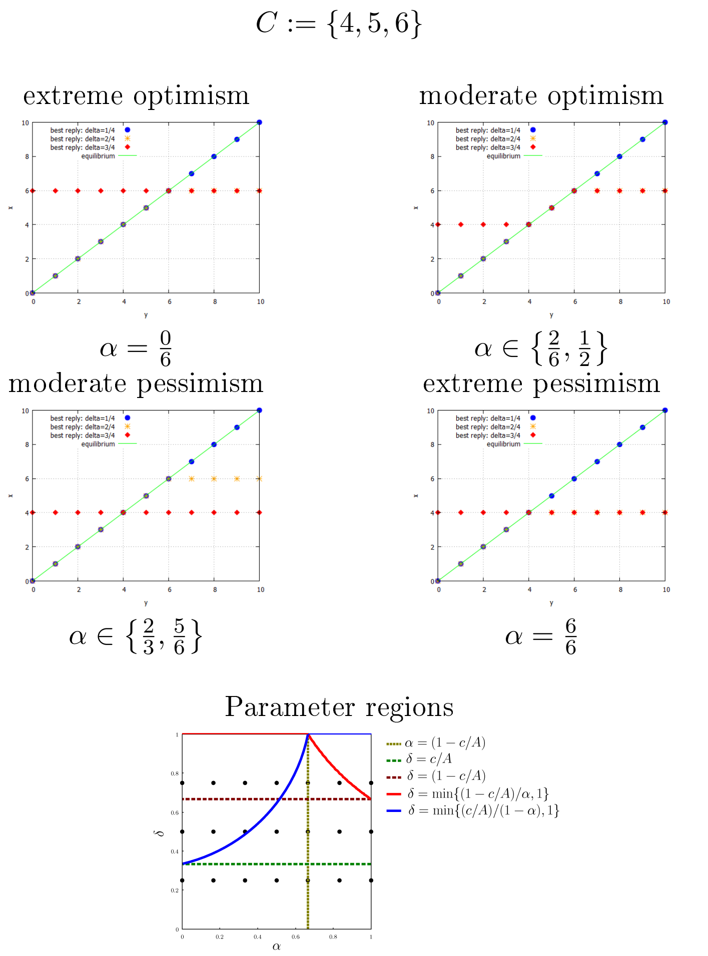

Figure 3: Best replies & equilibria for $A = 3$ and $c = 1$

the Nash equilibria for any degree of confidence. Obviously, such selection will depend on the degree ambiguity attitudes.

For the minimum-effort game with $A = 2c$, which is most commonly studied in the literature (see Huyck, Battalio, and Beil, 1990; Weber, 2006), the following Corollary 4.1 shows that, for any degree of ambiguity attitude α and any degree of confidence δ, there is a CD-EUA with a contribution level consistent with the context information C.

Corollary 4.1. *Consider a minimum-effort game with $A = 2c$. Let $\mathcal{M}(C, \alpha, \delta)$ be the set of symmetric CD-EUAs for $\alpha, \delta \in [0, 1]$ and context information C. Then,*

(i) if $\alpha > \frac{1}{2}$, then $\bigcap_{\delta \in [0,1]} \mathcal{M}(C, \alpha, \delta) := \{s^m\}$;

(ii) if $\alpha < \frac{1}{2}$, then $\bigcap_{\delta \in [0,1]} \mathcal{M}(C\, \alpha, \delta) := \{s^M\}$;

(iii) *if* $\alpha = \frac{1}{2}$, *then* $\bigcap_{\delta \in [0,1]} \mathcal{M}(C, \alpha, \delta) := \{s^y \in S \mid m \leq y \leq M\}$.

In this special case, pessimists ($\alpha > \frac{1}{2}$) choose the smallest contribution level s^m, while optimists ($\alpha < \frac{1}{2}$) will choose the highest contribution level s^M associated with context information C in any CD-EUA for any degree of confidence in the context information.

5 Games with context information

Substantial evidence, both from laboratory experiments and the field, shows that the information about players, strategy sets and payoffs provided in the formal description of a game does not suffice to predict the outcome of an interaction. Context information about the situation in which a game is played may, however, shape beliefs and, thus, influence behavior. In a review of Schelling's (1960) famous book, Myerson (2009) refers to his focal-point theory as

> "[...] *one of the most important ideas in social theory. Recognizing the fundamental social problem of selecting among multiple equilibria can help us to better understand the economic impact of culture on basic social phenomena such as social relationships, property and justice, political authority and legitimacy, foundations of social institutions, reputations and commitment, international boundaries in peace and war, and even the social use of the divine." (Myerson, 2009, p.1112)*

Nash equilibria represent stable beliefs constellations based on individual payoffs and individual rationality. Yet, there may be many such equilibrium beliefs constellations consistent for a particular game. By shaping players' beliefs, context information may induce a particular equilibrium from a set of multiple Nash equilibria.[35]

The concept of CD-EUA allows us to model how focal points and other context-dependent information may influence players' beliefs. In a CD-EUA, the equilibrium beliefs consists of two parts: an endogenous equilibrium belief about the behavior of the opponents and an exogenous belief about behavior gleaned from a game's context. The degree of confidence is the weight given to the exogenous context information. Context information may be specific in suggesting a particular behavior of the opponents or just put constraints on beliefs about the opponents' behavior. Complete ignorance is the extreme case where there is no information available about any specific behavior and, hence, there is pure ambiguity about the opponents' strategic behavior.

Remark 5.1 (strategic information). There is a large game-theoretic literature dealing with strategic transmission of information. The literature on mechanism design, signalling games, voting games, and part of contract theory is especially concerned with the

[35] Rather than appealing to context information as shaping beliefs, one may consider context-dependent payoff functions in order to select particular equilibria. Myerson (2009) argues convincingly for cultural and social environments influencing equilibrium selection through context information rather than modified preferences.

issue of strategically manipulating information in order to further some players' interests. Context information, as we study it here, is not supposed to be of this type. Context information is assumed to be non-strategic. Whether this assumption can be justified or whether one needs to consider strategic information manipulation has to be argued for in a given application.

Clearly, there is context information in the sense of (Schelling, 1960) such as "people drive on the left side of the road" which cannot be manipulated easily because it can be checked quickly and it concerns a large number of players. In other cases, credibility may strongly depend on the source of the context information. For example, a statement that "people donated x% of their income", is less likely to be strategic if published in a general newspaper than if written in a fund raising appeal. In general, context information about the behavior of a particular group of players is more likely to be strategically manipulated if supplied by a source close to this group.

It appears natural to assume that confidence changes in response to changing information in a sequence of plays. It is, however, beyond the scope of this paper to investigate such dynamics in detail. As ambiguity decreases, confidence in the players' own beliefs grows and leads eventually to a Nash equilibrium induced by the context information (Corollary 3.1). In this way, context information may facilitate selection among multiple Nash equilibria.

5.1 Specific context information

Even in the controlled environment of laboratory experiments, games are played by individuals with prior experience from interactions in similar situations. Moreover, different individuals may interpret the information about the game structure (strategy sets and payoff functions) provided by the experimenter in different ways. Laboratory experiments provide ample evidence for the importance of framing of the observed behavior in interactive situations (e.g., see Camerer, 2003; Dufwenberg, Gächter, and Hennig-Schmidt, 2011; Hopfensitz and Lori, 2016).

Context information in social interactive situations often provides clues for the way people interact. Such information may derive from conventions, e.g., for driving on a particular country-specific side of the road, or from rules about behavior, e.g., priority rules when passing a narrow bridge.[36] The following examples shall illustrate how specific context information may shape beliefs and, hence, induce particular behavior in a CD-EUA:

- interpreting behavior as reflecting *personal relationships* (Myerson, 2009),
- information about *past behavior*, *status-quo strategies* (Weber, 2006), and
- information about *bank withdrawals* (Brown, Trautmann, and Vlahu, 2017) .

[36]Whether a particular rule or convention is just a coordinating device or will become the source of strategic behavior will depend as well on the payoffs associated with these rules.

Finally, we will argue that the case of strategic behavior under complete ambiguity, that is "without any specific information", can be viewed as a special case of context information reflecting complete ignorance. Indeed, this case has been studied most intensively in the economic literature so far.

5.2 Personal relationships

Myerson (2009) illustrates the logic of focal points by labeling strategies in a co-ordination game. Referring to *personal relationships* serves to characterize mutually beneficial strategies as "friendly" and mutually harmful ones as "aggressive" in the Stag-hunt game:[37]

		Player 2	
		2 friendly	2 aggressive
Player 1	1 friendly	50,50	0,40
	1 aggressive	40,0	20,20

Labeling strategies may convey probabilistic context information of "friendly" behavior by the mass distributions: $\beta_1(\{2 \text{ friendly}\}) = 1$ and $\beta_2(\{1 \text{ friendly}\}) = 1$ or, alternatively, for "aggressive" behavior, by the mass distributions $\beta_1(\{2 \text{ aggressive}\}) = 1$ and $\beta_2(\{1 \text{ aggressive}\}) = 1$.

Suppose that both players are extreme pessimists with $\alpha_1 = \alpha_2 = 1$. For low degrees of confidence in the labeling, $\delta_1, \delta_2 < \frac{2}{3}$, there are two CD-EUA, ($\hat{\sigma}_1(\{2 \text{ friendly}\}) = 1$, $\hat{\sigma}_2(\{1 \text{ friendly}\}) = 1$) and ($\hat{\sigma}_1(\{2 \text{ aggressive}\}) = 1$, $\hat{\sigma}_2(\{1 \text{ aggressive}\}) = 1$), corresponding to the two Nash equilibria in pure strategies. For high degrees of confidence in the labeling, $\delta_1, \delta_2 > \frac{2}{3}$, however, there is a unique CD-EUA ($\hat{\sigma}_1(\{2 \text{ friendly}\}) = 1, \hat{\sigma}_2(\{1 \text{ friendly}\}) = 1$) where players coordinate on the strategies $(1 \text{ friendly}, 2 \text{ friendly})$.

Notice that the critical values for δ_1, δ_2 depend on the payoffs of the players and, in general, on their degrees of optimism and pessimism (here $\alpha_1 = \alpha_2 = 1$). For $u_1(f, a) - u_1(a, a)$ close to 0, a small degree of belief in a friendly context will suffice to make $(1 \text{ friendly}, 2 \text{ friendly})$ the unique CD-EUA. The difference measures the "cost of not co-ordinating".

5.3 Status-quo strategies

Huyck, Battalio, and Beil (1990) conducted a series of experiments on the minimum-effort (or weakest-link) game. A striking observation of their study is the successful co-ordination on the highest contribution level for small groups (2 players) and the consistent failure to coordinate on any other level but the smallest contribution for large groups (15 players). Weber (2006) implemented a variant of the game and investigated how information about play in previous rounds can serve as a focal point for co-ordination.

In Weber (2006) players had to choose one of several integer contribution levels ranging from the lowest contribution level $s^{\min} = 1$ to the highest contribution of $s^{\max} = 7$. Hence, $S_i := T = \{1, \ldots, 7\}$. The experiment begins with small groups of players and

[37] The game structure is analogue to Example 3.2.

increases the set of players successively by new entrants. In a *no-history treatment*, no information about the other players behavior was released. In the *history treatment*, previous minimal contributions were made public; i.e., "entrants learned the group's history (the minima obtained in all previous periods) and this was common knowledge." (Weber, 2006, p.118).

A stylized summary of the behavior observed by Weber (2006) notes the following features regarding the information about the observed minimal contributions.

1. Without information, in the *no-history treatment*, co-ordination on the highest contribution $s^{\max}$ was achieved only for a group of two players. As in Huyck, Battalio, and Beil (1990), with a growing number of players contribution levels did collapse quite rapidly to the "safe," lowest contribution level s^{min}.

2. In the history treatment, one could observe the following behavior. For groups of two players, co-ordination on the highest contribution level $s^{\max}$ could be achieved. With a growing number of players, this high level of contributions could be maintained for several periods until group-size had grown to between 4 and 6 players. As group size increased further, the minimal contributions would fall and converge to the lowest level $s^{\min}$, sometimes collapsing rapidly, at other times slowly, stabilizing intermittently at some *intermediate* contribution level.

We would like to argue that information about the record of co-ordination in a group provides "context information" which is relevant for the choice of strategy but which is not contained in the description of the game and, hence, not considered in a Nash equilibrium.

Denoted by $m \in T$ the minimum contribution level that has been observed. Suppose players learn that the group had achieved this minimum effort level m. It seems natural to assume that these players will give some extra weight to the set of strategy combinations of the opponents yielding a minimum contribution level of at least m.[38] Let $\underline{m}(s_{-i}) := \min\{s_j \in \langle s_{-i}\rangle : j \neq i\}$ denote the lowest contribution of the opponents in the strategy combination s_{-i}. We denote by $S_{-i}(m) := \{s_{-i} \in S_{-i} : m \leq \underline{m}(s_{-i})\}$ be the set of the opponents' strategy combinations yielding a minimum contribution of at least m.[39]

To take into account m, a player i's context information may assign a higher weight to $S_{-i}(m)$ than to other events. Let $\beta_i(m)$ denote a mass distributions on Σ_{-i} such that $\beta_i(S_{-i}(m)) = 1$ and $\beta_i(E) = 0$ for any other E in Σ_{-i}. Since $S_{-i}(m)$ is a non-singleton, $\beta_i(m)$ reflects ambiguity about the strategies yielding a contribution of at least m.[40]

The following result shows that for sufficiently high degrees of confidence in the context information of a minimum contribution m, in any CD-EUA, players do not contribute less than m, regardless of their degrees of ambiguity attitude.

[38]For simplicity of exposition, we do not distinguish between entrants and incumbents. In principle, one may use differences in beliefs between an entrant and former group members to model the dynamics observed in the experiments by Weber (2006).

[39]Notice that $S_{-i}(m)$ is more general than C_{-i} for $C = \{m, \ldots, 7\}$, the case we studied in Section 4.

[40]In Weber (2006), payoffs were given by $u_i(s_i, s_{-i}) := A \min\{s_1, \ldots, s_n\} - cs_i + d$ with $A > c > 0$. For the experimental study, they chose $n = 7$, $a = 0.2$, $c = 0.1$ and $d = 0.2$.

Proposition 5.1. *Let $\alpha = (\alpha_1, \dots, \alpha_n) \in [0,1)^n$ be a vector of degrees of ambiguity attitudes, and $\beta(m) := (\beta_1(m), \dots, \beta_n(m))$ be mass distributions incorporating context information m. Let $\delta = (\delta_1, \dots, \delta_n) \in [0,1)^n$ be a vector of confidence parameters, each one satisfying*

$$\delta_i \geq \overline{\delta}(m) := \frac{A\left(m - s^{\min}\right) - (A - c)}{A\left(m - s^{\min}\right)} \quad \text{for each } i \in I. \tag{5.1}$$

Then, any CD-EUA $\left(\phi_1^{\gamma_1(\sigma_1(\hat{\tau}),\beta_1(m),\delta_1)}, \dots, \phi_n^{\gamma_n(\sigma_n(\hat{\tau}),\beta_n(m),\delta_n)}\right)$ will have minimal contributions of at least m. That is, $s_j \geq \underline{m}(s_{-j})$ for all $s_j \in \langle s_{-i} \rangle$ with $s_{-i} \in \operatorname{supp} \sigma_i(\hat{\tau})$ and all $i \in I$.
Moreover, if $m = s_i^{\max}$ holds for all $i \in I$, then $\operatorname{supp} \sigma_i(\hat{\tau}) = \{s_{-i} \in \Sigma_{-i} | \ \underline{m}(s_{-i}) = m\}$.

Proof. See Appendix 7. □

The result of Proposition 5.1 shows that sufficient confidence in the context information induces equilibrium beliefs which rule out CD-EUA with contributions below m. Notice that the critical degrees of confidence $\overline{\delta}$ depends on the minimal level of previously observed contributions m relative to the worst contribution level $s^{\min}$. The smaller the previously observed contribution level m relative to the minimal effort level, $s^{\min}$, the more confidence in this information is required in order to maintain the level of contributions.[41] The highest level of confidence is required to achieve the maximal contribution level.

Notice that the critical level $\overline{\delta}(m)$ is a decreasing function of m. Although the degree of confidence is a personal characteristic of the player which one cannot observe directly, comparative static predictions regarding CD-EAU can be tested.[42]

5.4 Bank-run game

Brown, Trautmann, and Vlahu (2017) study possible channels of contagion among banks in an experiment with a simplified version of the bank-run model of Diamond and Dybvig (1983).[43] In particular, they study two banks where depositors choose their withdrawals independently but sequentially. Depositors of the bank that moved first were informed about the quality of their bank's assets while depositors of the second bank were uncertain about the quality of their bank's asset.[44] The withdrawal behavior of depositors of the bank moving first, the *leader,* would be communicated to the depositors of the second bank, the *follower,* before their withdrawal decision. Although there was no direct link between the assets of the two banks, depositors in the second bank could condition their

[41] When comparing this result with the more general analysis in Section 4 (Proposition 4.2), note that $\overline{\delta}(m) = \frac{(m-s^{\min})-1}{(m-s^{\min})} + \frac{1}{(m-s^{\min})}\frac{c}{A} > \frac{c}{A}$ for $A - c > 1$ and $\left(m - s^{\min}\right) > 1$. Hence, $\delta \geq \overline{\delta}(m)$ implies $\delta \geq \frac{c}{A}$, i.e., case (i) of Lemma 7.1 in Appendix 7.

[42] Compare Eichberger, Kelsey, and Schipper (2009) for an experimental study testing comparative static behavior based on critical values of the degree of ambiguity.

[43] We are grateful to Stefan Trautmann who suggested this model to us as an application of CD-EUA.

[44] The quality of a banks asset did, however, not affect the payouts of withdrawals.

beliefs on the information about the withdrawals in the first bank.[45] In some treatments (*linkage*), depositors of the second bank were told that both banks had the same asset quality. Brown, Trautmann, and Vlahu (2017, p.3) describe their setup as follows:

> "The underlying game in our experimental design is based on a two-person co-ordination game. In this game there are two depositors, depositor A and depositor B, in a bank. Both depositors decide simultaneously whether to keep their deposit in the bank until maturity, or to withdraw their funds. If both depositors keep their funds in the bank, the bank does not have to liquidate any investments and both depositors receive a payoff R. If either depositor withdraws the deposit the bank is liquidated. We assume that the liquidation value of the bank's investment is L. If only one depositor withdraws, that depositor receives a payoff of L and the other depositor receives 0. If both depositors withdraw, each receives a payoff of $L/2$."

		Depositor B	
		$k(eep\ deposit)$	$w(ithdraw)$
Depositor A	$k(eep\ deposit)$	R_t, R_t	$0, L$
	$w(ithdraw)$	$L, 0$	$\frac{L}{2}, \frac{L}{2}$

with $t \in T := \{w(eak), s(trong)\}$.

The information of the depositors regarding the follower bank is described by Brown, Trautmann, and Vlahu (2017) as follows. Before choosing whether to withdraw or not, depositors of the follower bank would learn the result of the leaders' game, i.e., the number of withdrawals that occurred - $0, 1$, or 2. The information about the number of withdrawals from the leading bank can be viewed as context information that is common knowledge for both depositors in the follower bank. Table 1 shows the mass distributions of depositors $i \in \{A, B\}$ derived from the available information about withdrawal behavior in the leading bank.

withdrawals	public information	mass distribution $i \in \{A, B\}$
0	$\beta(\{(k,k)\}) = 1$	$\beta_i(\{k\}) = 1$
1	$\beta(\{(k,w),(w,k)\}) = 1$	$\beta_i(\{k,w\}) = 1$
2	$\beta(\{(w,w)\}) = 1$	$\beta_i(\{w\}) = 1$

Table 1: Context information about the leading bank

The treatments "linkage" (l) or "no linkage" (nl) are likely to affect the degree of relevance of this context information, i.e., $\delta(nl) < \delta(l)$.

Assuming w.l.o.g. $u(x) = x$, the CEU payoff of Depositor $i = A, B$ is

$$V_i(k \mid \sigma_i, \beta_i, \delta_i(t)) = \left(1 - \delta_i(t)\right)\left[\sigma_i(\{k\})R\right] + \delta_i(t)\left[\beta_i(\{k\})R + \beta_i(\{k,w\})\left((1-\alpha_i)R\right)\right]$$

[45] In a similar experiment, Chakravarty, Fonseca, and Kaplan (2014) use a design that is even closer to the original model of Diamond and Dybvig (1983).

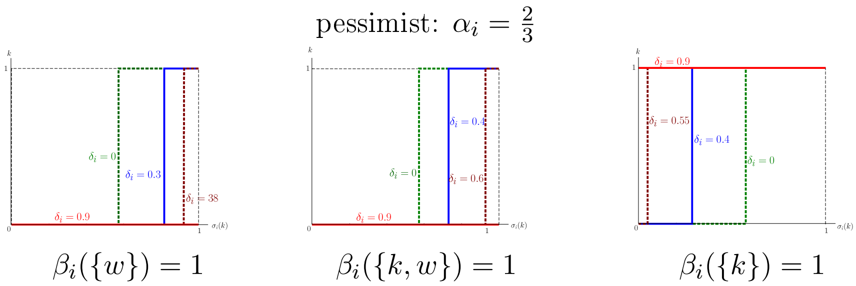

Figure 4: Best replies of Depositor A

if the depositor keeps her deposit, k, and

$$V_i(w \mid \sigma_i, \beta_i, \delta_i(t)) = (1 - \delta_i(t)) \left[\sigma_i(\{k\})L + \sigma_i(\{w\})\frac{L}{2}\right] + \delta_i(t) \left[\beta_i(\{k\})L + \beta_i(\{w\})\frac{L}{2} + \beta_i(\{k, w\}) \left(\alpha_i \frac{L}{2} + (1 - \alpha_i)L\right)\right].$$

if she withdraws it, w. Depositor i will keep her funds in the bank, if

$$V_i\big(k \mid \sigma_i, \beta_i, \delta_i(t)\big) \geq V_i\big(w \mid \sigma_i, \beta_i, \delta_i(t)\big), \tag{5.2}$$

that is, if her beliefs about the other depositor keeping her deposit satisfy $\sigma_i(k) \geq \overline{\sigma}_i^k$ for the critical value $\overline{\sigma}_i^k(\overline{r}, L, \alpha_i, \delta_i(t), \beta_i)$ given by

$$\overline{\sigma}_i^k(\overline{r}, L, \alpha_i, \delta_i(t), \beta_i) :=$$

$$\frac{(1 - \delta_i(t))\frac{L}{2} + \delta_i(t)\left[\beta_i(k)\left(L - \overline{r}\right) + \beta_i(w)\frac{L}{2} + \beta_i(k, w)\left(\alpha_i \frac{L}{2} + (1 - \alpha_i)\left(L - \overline{r}\right)\right)\right]}{(1 - \delta_i(t))\left(\overline{r} - \frac{L}{2}\right)}$$

where $\overline{r} := \frac{1}{2}R_s + \frac{1}{2}R_w$ is the average asset return of the follower bank. The public context information allows for only three cases:

1. $\beta_A(\{k\}) = \beta_B(\{k\}) = 1$, if no withdrawals are reported from the leading bank,
2. $\beta_A(\{w\}) = \beta_B(\{w\}) = 1$, if two withdrawals are reported from the leading bank, or
3. $\beta_A(\{k, w\}) = \beta_B(\{k, w\}) = 1$ if only one withdrawal is reported from the leading bank.

The following diagram shows the best-reply correspondence for a depositor of the follower bank in each of these cases.

The left diagram of Figure 4 shows the best reply of a depositor for the case when both depositors of the leader bank had withdrawn their deposits, $\beta_i(\{w\}) = 1$. For no confidence in this information, $\delta_i = 0$, (the green best reply), CD-EUA coincides with one of the two Nash equilibria and, hence, both $\sigma_i(\{k\}) = 1$, i.e., $\sigma_i(\{w\}) = 0$, and $\sigma_i(\{k\}) = 0$, i.e., $\sigma_i(\{w\}) = 1$, remain possible equilibria. As δ_i increases, however, the best reply correspondence shifts to the right until, for $\delta_i \geq 0.58$, there remains a unique CD-EUA where all depositors withdraw their funds, $\sigma_i(\{k\}) = 0$ or $\sigma_i(\{w\}) = 1$ (the red best reply). Hence, a CD-EUA with context information that all depositors of the leading bank withdrew their deposits induces a CD-EUA for the follower bank where both players withdraw their deposits as well, provided their confidence in the information about the leader bank is sufficiently high.

A similar analysis shows that depositors will leave their deposits with the follower bank if they learn that depositors in the leading bank did not withdraw their deposits and if they trust this context information about the leader bank (see the right diagram of Figure 4. For these clear-cut cases, ambiguity attitude does not matter.

This is different, however, when only one withdrawal of deposits occurred in the leading bank. In this case ambiguity will matter. The central diagram of Figure 4 shows the respective best reply correspondences for pessimistic players ($\alpha_i = \frac{2}{3}$). Increasing confidence in context information will lead depositors to withdraw their deposits. The following Proposition summarizes the result illustrated in the diagrams.

Proposition 5.2. *For each case of context information, there degrees of confidence* $\overline{\delta}_A, \overline{\delta}_B \in (0,1)$ *such that there is a unique CD-EUA with the following properties:*

- *for* $\beta_A(\{k\}) = \beta_B(\{k\}) = 1$ *and* $\delta_A \geq \overline{\delta}_A^k$ *and* $\delta_B \geq \overline{\delta}_B^k$, *all depositors will keep their deposits in the follower bank:* $\hat{\sigma}_A(\{k\}) = \hat{\sigma}_B(\{k\}) = 1$;
- *for* $\beta_A(\{w\}) = \beta_B(\{w\}) = 1$ *and* $\delta_A \geq \overline{\delta}_A^w$ *and* $\delta_B \geq \overline{\delta}_B^w$, *all depositors will keep their deposits in the follower bank:* $\hat{\sigma}_A(\{w\}) = \hat{\sigma}_B(\{w\}) = 1$;
- *for* $\beta_A(\{k,w\}) = \beta_B(\{k,w\}) = 1$ *and* $\delta_A \geq \overline{\delta}_A^{kw}(\alpha_A)$ *and* $\delta_B \geq \overline{\delta}_B^{kw}(\alpha_B)$, *pessimistic depositors* ($\alpha_i > \frac{1}{2}$) *will withdraw their deposits in the follower bank:* $\hat{\sigma}_i(\{w\}) = 1$ *and optimistic depositors* ($\alpha_i < \frac{1}{2}$) *will keep their deposits with the follower bank,* $\hat{\sigma}_i(\{k\}) = 1$.

While context information β_i is common knowledge in this experiment, the degree of confidence or trust in it, δ_i, as well as the attitude towards ambiguity α_i are individual characteristics of the players. Figure 5 shows the relationship between the degree of confidence δ_i and the critical value $\overline{\sigma}_i(55, 40, \alpha_i, \delta_i, \beta_i)$ of depositors for keeping her deposits with the bank. For information that the other bank's depositors withdrew all deposits (the blue curve), the critical threshold for keeping deposits in the bank $\overline{\sigma}_i$ increases rapidly to 1. Hence, withdrawing deposits becomes a dominant strategy for players with a degree of confidence above the critical level $\overline{\delta}_i$. For $\beta_i(\{k\}) = 1$ (the green curve) the critical threshold will fall to 0 and all depositors with higher degrees of confidence will keep their deposits. The yellow and red curves represent cases of ambiguity,

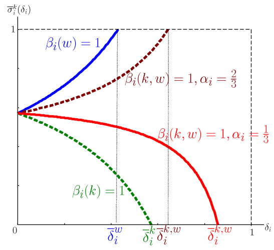

Figure 5: Critical values of beliefs: $\overline{\sigma}_i^{\{k\}}(\delta_i)$

$\beta_i(\{k, w\}) = 1$, where the ultimate dominant strategy depends on the attitude towards ambiguity α_i.

Although the degree of confidence is an individual characteristic, one may assume that this characteristic is distributed over a population. In this case, one can interpret the critical values $\overline{\delta}_i$ as indicating the proportion of a population withdrawing in response to context information. From this perspective, the "linkage" information is likely to increase the relevance of the withdrawal information, $\delta(nl) < \delta(l)$, and thus to re-enforce the effects of context information.

The experimental results reported in Brown, Trautmann, and Vlahu (2017, Table 2) confirm the impact of the context information. Observing withdrawals in the leading bank increases withdrawals in the follower bank significantly. Moreover, these effects were consistently stronger in the linkage treatment.

5.5 No context information

The types of context information discussed so far did shape the beliefs of players and, thus, behavior in an CD-EUA. Context information may be very specific. In some cases, context information may even be probabilistic suggesting a particular behavior of players as in Subsections 5.2 and 5.4. Sometimes context information may be non-probabilistic and only impose certain constraints on beliefs as in Subsections 5.3. In other cases, however, there may be no specific context information at all. Yet, players still may have more or less confidence in their endogenous probabilistic equilibrium beliefs. If they feel strong ambiguity about their opponents' strategic behavior without a clue from the context of the game regarding the other players behavior, this suggests strong ambiguity about the likely behavior of the other players.

If players have no specific context information, then no strategy combination of the opponents can be excluded based on context information. In such a situation, the lack of exogenous information will be captured by the mass distribution $\beta_i(\{S_{-i}\}) = 1$ for all players $i \in I$, reflecting complete ignorance regarding the behavior of the opponents (see Examples 2.2, 3.2 and 3.1). The mass distribution for such context information is

$$\gamma_i(E \mid \sigma_i, \beta_i, \delta_i) = \begin{cases} (1 - \delta_i)\sigma_i(\{s_{-i}\}) & \text{if} \quad E = \{s_{-i}\}, \\ \delta_i & \text{if} \quad E = S_{-i}, \\ 0 & \text{otherwise}, \end{cases} \tag{5.3}$$

for each $E \in \Sigma_{-i}$ and the associate belief function is a simple capacity (see Example 2.1). For pure pessimism, simple capacities have been often applied to games with strategic ambiguity both in experimental studies and in economic applications (e.g., Eichberger and Kelsey, 2002, 2011). In case of complete ignorance, however, the degrees of confidence regarding Nash equilibrium play, together with players' attitudes towards ambiguity, that is their optimism or pessimism will determine equilibrium behavior. In this special case, the CD-EUA and the EAU of Eichberger and Kelsey (2014) provide the same equilibrium predictions (see Lemma 10.2, Appendix 10).

6 Conclusion

In this paper, we presented a new solution concept for strategic games with exogenous information about players' behavior which is not part of the description of the game. To capture the idea that equilibrium beliefs may be influenced by a "context of a game", we introduced the notion of context-dependent belief functions to model players' beliefs. Belief functions with their (additive) mass distributions are a useful tool which allows us to include into players' beliefs various types of partial or incomplete information about the other players behavior.

The concept of Context-Dependent Equilibrium under Ambiguity (CD-EUA) offers the possibility to study how "contextual details" in combination with payoff-based information will shape the equilibrium behavior. We showed that one can include aspects of a game's environment and examine whether the corresponding CD-EUA will support a particular Nash equilibrium, helping us to select among multiple equilibria based on context information, or to what extend equilibrium behavior will depart from the Nash equilibrium predictions. Thus, in the spirit of Schelling (1960), the CD-EUA may help to improve the applicability of game-theoretic analysis.

We could prove existence of a CD-EUA for any finite normal-form game, arbitrary degrees of confidence regarding context information, and arbitrary degrees of optimism and pessimism. Moreover, we could show that the CD-EUA equilibrium correspondence is upper hemi-continuous with respect to the players' degrees of confidence. Thus, CD-EUA converges to the set of Nash equilibria as players become fully confident that their equilibrium beliefs correctly represent the probabilities of the opponents' optimal strategy choices. The generality of these results together with tractability of the CD-EUA concept allow for a wide range of applications in game theory and economics.

The types of context information which CD-EUA can accommodate range from a very precise probabilistic statement that players behave according to a certain strategy combination to situations of great ambiguity reflected in complete ignorance. In this sense, CD-EUA provide a unifying view about equilibria in games ranging from Nash equilibrium at one extreme to EUA in the spirit of Eichberger and Kelsey (2014) and Marinacci (2000) when there is strategic ambiguity yet no specific information about the opponents' behavior from the context of the game.

It is conceivable that some types of context information and equilibrium behavior may be correlated. For instance, probabilistic context information may be represented by a joint probability distribution which suggests the extension of CD-EUA to a correlated equilibrium notion. It is beyond the scope of this paper to investigate this possibility in some depth. The careful study of correlated equilibria in Forges (2006) provides a natural point of departure for such an extension. Indeed, context information may concern a common signal or a mediation process leading to a correlated equilibrium as ambiguity vanishes. The paper discusses several examples, e.g., in Section 5.3, where context information induces behavior which is correlated. The assumption of independence for the endogenous part of beliefs guarantees, however, that such correlations in behavior will vanish the more confident players become of their endogenous equilibrium beliefs. Due to the independence of the equilibrium beliefs, we could prove that CD-EUA would converge to Nash equilibria as confidence in the exogenous context information vanishes (Proposition 3.2). It is left for future research to investigate if, and under what conditions, an extended notion of CD-EUA will approximate correlated equilibria as ambiguity disappears.

It also appears natural to assume that confidence changes in response to observed behavior in a sequence of plays, similar to Eichberger and Guerdjikova (2018). This possible extension is, however, beyond the scope of this paper and, hence, left for future research. As ambiguity decreases with observations, confidence in the endogenous part of beliefs grows and leads eventually to a Nash or correlated equilibrium induced by the context information.

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7 Proofs

7.1 Proof of Theorem 3.1

Proof. Consider a game Γ_β with context-related mass distributions $\beta = (\beta_1, \ldots, \beta_n)$, i.e., with $\beta_i \in \Delta(\Sigma_{-i})$ for all $i \in I$. Fix the ambiguity attitudes $(\alpha_1, ..., \alpha_n) \in [0,1]^n$ and the vector of confidence parameters $\delta = (\delta_1, ..., \delta_n) \in [0,1)^n$. For $\tau = (\tau_1, \ldots, \tau_n) \in \times_{i \in I} \Delta(S_i)$, define the probability distribution $\sigma_i(\tau)$ on Σ_{-i} as a function of τ by

$$\sigma_i(\{s_{-i}\} \,|\tau) := \prod_{s_j \in \langle s_{-i} \rangle} \tau_j(s_j) \tag{7.1}$$

for each singleton event $\{s_{-i}\} \in S(\Sigma_{-i})$ and $\sigma_i(E|\tau) = 0$ for all other events $E \in N(\Sigma_{-i})$. Using $\sigma_i(\tau), \beta_i$ and δ, define the mass distribution $\gamma_i(\sigma_i(\tau), \beta_i, \delta_i)$ on Σ_{-i} by

$$\gamma_i(\sigma_i(\tau), \beta_i, \delta_i)(E) := (1 - \delta_i)\sigma_i(E|\tau) + \delta_i \beta_i(E) \tag{7.2}$$

for each $E \in \Sigma_{-i}$. The associated belief function $\phi_i^{\gamma_i(\sigma_i(\tau),\beta_i,\delta_i)}$ on Σ_{-i} is given by

$$\phi_i^{\gamma_i(\sigma_i(\tau),\beta_i,\delta_i)}(E) = \sum_{F \subseteq E} \gamma_i(F \mid \sigma_i(\tau), \beta_i, \delta_i) \tag{7.3}$$

for each $E \in \Sigma_{-i}$. By Proposition 2.1, the Choquet expected payoff of a $s_i \in S_i$ is

$$\begin{aligned} V_i\big(s_i \mid \alpha_i, \phi_i^{\gamma_i(\sigma_i(\tau),\beta_i,\delta_i)}\big) &:= V_i\big(s_i \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i, \delta_i)\big) \\ &= \sum_{E \subseteq S_{-i}} \gamma_i(E \mid \sigma_i(\tau), \beta_i, \delta_i) V_i^{\alpha_i}(s_i, E) \\ &= (1 - \delta_i) \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \sigma_i(\{s_{-i}\}) + \delta_i \sum_{E \subseteq S_{-i}} \beta_i(E) V_i^{\alpha_i}(s_i, E) \end{aligned} \tag{7.4}$$

where

$$V_i^{\alpha_i}(s_i, E) := \left[\alpha_i \min_{s_{-i} \in E} u_i(s_i, s_{-i}) + (1 - \alpha_i) \max_{s_{-i} \in E} u_i(s_i, s_{-i})\right] \tag{7.5}$$

For all $i \in I$, define a mapping $W_i : \Delta(S_1) \times \ldots \times \Delta(S_n) \longrightarrow \mathbb{R}$ as

$$W_i(\tau) := \sum_{s_i \in S_i} \tau_i(s_i) V_i\big(s_i \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i, \delta_i)\big). \tag{7.6}$$

$W_i(\tau)$ is a continuous function on $\Delta(S_1) \times \ldots \times \Delta(S_n)$ and linear in τ_i.

Now, define $B_i : \Delta(S_1) \times \ldots \times \Delta(S_n) \rightrightarrows \Delta(S_i)$ to be

$$B_i(\tau) := \arg \max_{\tau_i \in \Delta(S_i)} W_i(\tau), \tag{7.7}$$

the correspondence of maximizers of $W_i(\tau)$ with respect to τ_i, a probability distribution on S_i. By the Maximum Theorem, the correspondence $B_i : \Delta(S_1) \times \ldots \times \Delta(S_n) \rightrightarrows \Delta(S_i)$

is non-empty, upper hemi-continuous and convex-valued since $W_i(\tau)$ is a continuous function on the compact set $\Delta(S_1) \times \ldots \times \Delta(S_n)$ and linear in τ_i.

The Cartesian product of n correspondences $B(\tau) := B_1(\tau) \times \ldots \times B_n(\tau)$ is a correspondence $B : \Delta(S_1) \times \ldots \times \Delta(S_n) \rightrightarrows \Delta(S_1) \times ... \times \Delta(S_n)$ which inherits the properties of its components. Hence, all conditions of the Kakutani Fixed Point Theorem are satisfied. Hence, there exists $\widehat{\tau} \in \Delta(S_1) \times \ldots \times \Delta(S_n)$ such that $\widehat{\tau} \in B(\widehat{\tau})$.

Given $\widehat{\tau}$, it remains to be shown that the induced n-tuple of belief functions

$$(\phi_1^{\gamma_1(\sigma_1(\hat{\tau}),\beta_1,\delta_1)}, \ldots, \phi_n^{\gamma_n(\sigma_n(\hat{\tau}),\beta_n,\delta_n)}) \tag{7.8}$$

defined by the mass distributions $(\gamma_1(\sigma_1(\widehat{\tau}), \beta_1, \delta_1), \ldots, \gamma_n(\sigma_n(\widehat{\tau}), \beta_n, \delta_n))$ are an CD-EUA.

For each $i \in I$, $s_{-i} \in \operatorname{supp} \sigma_i(\hat{\tau})$ if and only if $\sigma_i(\{s_{-i}\} \,|\widehat{\tau}) := \prod_{s_j \in \langle s_{-i} \rangle} \widehat{\tau}_j(s_j) > 0$. Hence, $\widehat{\tau}_j(s_j) > 0$ for all $s_j \in \langle s_{-i} \rangle$ and thus $s_j \in$ supp$(\widehat{\tau}_j)$.

Since $\widehat{\tau}_j \in B_i(\widehat{\tau})$,

$$V_j(s_j \mid \alpha_j, \gamma_j(\sigma_j(\hat{\tau}), \beta_j, \delta_j)) \geq V_j(s'_j \mid \alpha_j, \gamma_j(\sigma_j(\hat{\tau}), \beta_j, \delta_j)) \tag{7.9}$$

for all $s'_j \in S_j$. Hence, $s_j \in \text{supp}(\widehat{\tau}_j)$ implies that

$$s_j \in R_j\big(\sigma_j(\widehat{\tau}), \beta_j, \delta_j\big) := \arg\max_{s_j \in S_j} V_j\big(s_j \mid \alpha_j, \gamma_j(\sigma_j(\widehat{\tau}), \beta_j, \delta_j)\big) \tag{7.10}$$

and thus we have

$$\text{supp}\,\big(\sigma_j(\hat{\tau})\big) \subseteq \mathop{\times}_{j \neq i} R_j(\sigma_j(\widehat{\tau}), \beta_j, \delta_j), \tag{7.11}$$

completing the proof. □

7.2 Proof of Proposition 3.1

Proof. Consider a vector of probability distributions $\tau = (\tau_1, \ldots, \tau_n) \in \times_{i \in I} \Delta(S_i)$. For $(\delta_1, \ldots, \delta_n) = \mathbf{0}$, one has for all players $i \in I$ and all strategy combinations $s_{-i} \in S_{-i}$,

$$\gamma_i(\{s_{-i}\} \mid \sigma_i(\tau), \beta_i, 0) = \sigma_i(\{s_{-i}\} \mid \tau) := \prod_{s_j \in \langle s_{-i} \rangle} \tau_j(s_j). \tag{7.12}$$

Moreover, by Proposition 2.1, the CEU payoff of a strategy $s_i \in S_i$ is

$$\begin{aligned} V_i\big(s_i \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i, 0)\big) &= \sum_{\{s_{-i}\} \subseteq \Sigma_{-i}} \gamma_i(\{s_{-i}\}|\sigma_i(\tau), \beta_i, 0) \cdot V_i^{\alpha}(s_i, \{s_{-i}\}) \\ &= \sum_{s_{-i} \in S_{-i}} \sigma_i(\{s_{-i}\} \mid \tau) u_i(s_i, s_{-i}) \\ &= \sum_{s_{-i} \in S_{-i}} \Big(\prod_{s_j \in \langle s_{-i} \rangle} \tau_j(s_j) \Big) u_i(s_i, s_{-i}) := \mathcal{E}(s_i \mid \sigma_i(\hat{\tau})), \end{aligned} \tag{7.13}$$

where $\mathcal{E}(s_i \mid \sigma_i(\hat{\tau}))$ denotes the expected utility of s_i with respect to the product measure $\sigma_i(\tau)$.

(i) Consider a vector of probability distributions $\hat{\tau} = (\hat{\tau}_1, \ldots, \hat{\tau}_n) \in \times_{i\in I}\Delta(S_i)$ such that the n-tuple of belief functions

$$\left(\phi_1^{\gamma_1(\sigma_1(\hat{\tau}),\beta_1,0)}, \ldots, \phi_n^{\gamma_n(\sigma_n(\hat{\tau}),\beta_n,0)}\right), \tag{7.14}$$

induced by the mass distributions $\left(\gamma_1(\sigma_1(\hat{\tau}), \beta_1, 0), \ldots, \gamma_n(\sigma_n(\hat{\tau}), \beta_n, 0)\right)$ as in (7.12) constitutes a CD-EUA for the confidence vector $\delta = (\delta_1, \ldots, \delta_n) = \mathbf{0}$.

We show that $\widehat{\tau}$ must be a Nash equilibrium. By Definition 3.2, for all $i \in I$,

$$\varnothing \neq \operatorname{supp}(\sigma_i(\hat{\tau})) \subseteq \underset{j\neq i}{\times} R_j(\widehat{\sigma}_j, \beta_j, 0) \tag{7.15}$$

where

$$R_j(\widehat{\sigma}_j, \beta_j, 0) = \arg\max_{s_j\in S_j} V_j(s_j \mid \alpha_j, \gamma_j(\sigma_j(\widehat{\tau}), \beta_j, 0)), \tag{7.16}$$

implying that

$$\operatorname{supp}(\sigma_i(\hat{\tau})) \subseteq \underset{j\neq i}{\times} \arg\max_{s_j\in S_j} \mathcal{E}(s_j \mid \sigma_j(\hat{\tau})). \tag{7.17}$$

Hence, we have

$$\operatorname{supp}(\hat{\tau}_j) \subseteq \underset{j\neq i}{\times} \arg\max_{s_j\in S_j} \mathcal{E}(s_j \mid \sigma_j(\hat{\tau})), \tag{7.18}$$

showing that $\widehat{\tau}$ is a Nash equilibrium.

(ii) If $\hat{\tau} = (\hat{\tau}_1, \ldots, \hat{\tau}_n) \in \underset{i\in I}{\times} \Delta(S_i)$ is a Nash equilibrium, then for all $j \in I$,

$$\operatorname{supp}(\hat{\tau}_j) \subseteq \arg\max_{s_j\in S_j} \mathcal{E}(s_j \mid \sigma_j(\hat{\tau})). \tag{7.19}$$

Define $\left(\gamma_1(\sigma_1(\hat{\tau}), \beta_1, 0), \ldots, \gamma_n(\sigma_n(\hat{\tau}), \beta_n, 0)\right)$ as in (7.12). Hence, we have for all $i \in I$,

$$\begin{aligned}
\operatorname{supp}(\sigma_i(\widehat{\tau})) &\in \underset{j\neq i}{\times} \arg\max_{s_j\in S_j} \mathcal{E}(s_j \mid \sigma_j(\hat{\tau})) \\
&= \underset{j\neq i}{\times} \arg\max_{s_j\in S_j} V_j(s_j \mid \alpha_j, \gamma_j(\sigma_j(\widehat{\tau}), \beta_j, 0)) \\
&= \underset{j\neq i}{\times} R_j(\widehat{\sigma}_j, \beta_j, 0).
\end{aligned} \tag{7.20}$$

Hence, the corresponding n-tuple of belief functions

$$\left(\phi_1^{\gamma_1(\sigma_1(\hat{\tau})\beta_1,0)}, \ldots, \phi_n^{\gamma_n(\sigma_n(\hat{\tau}),\beta_n,0)}\right) \tag{7.21}$$

is a CD-EUA for $\delta = (\delta_1, \ldots, \delta_n) = \mathbf{0}$, completing the proof. □

7.3 Proof of Proposition 3.2

Proof. By Theorem 3.1, for each vector $\delta = (\delta_1, \dots, \delta_n) \in [0,1)^n$, $\Phi(\delta)$ is not empty. Consider a sequence $\delta^\lambda = (\delta_1^\lambda, \dots, \delta_n^\lambda) \in [0,1)^n$ that converges to $\mathbf{0}$, i.e., $\lim_{\lambda\to\infty} \delta^\lambda \to \mathbf{0}$.

Any sequence $\hat{\tau}^\lambda = (\hat{\tau}_1^\lambda, \dots, \hat{\tau}_n^\lambda) \in \Phi(\delta^\lambda) \subseteq \times_{i\in I}\Delta(S_i)$ is bounded. By the Bolzano-Weierstrass theorem, the sequence $\hat{\tau}^\lambda \in \Phi(\delta^\lambda)$ contains a converging subsequence $\hat{\tau}^\kappa$ (maintaining the superscript for simplicity). Since $\hat{\tau}^\lambda \in \Phi(\delta^\lambda)$, we have for all $i \in I$

$$\varnothing \neq \text{supp}\left(\phi^{\gamma_j(\sigma_j(\widehat{\tau}^\lambda),\beta_j,\delta^\lambda)}\right) \subseteq \mathop{\times}_{j\neq i} \arg\max_{s_j\in S_j} V_j\big(s_j \mid \alpha_j, \gamma_j(\sigma_j(\widehat{\tau}^\lambda), \beta_j, \delta^\lambda)\big) \tag{7.22}$$

which is equivalent to

$$\text{supp}\left(\sigma_j(\widehat{\tau}^\lambda)\right) \subseteq \mathop{\times}_{j\neq i} \arg\max_{s_j\in S_j} V_j\big(s_j \mid \alpha_j, \gamma_j(\sigma_j(\widehat{\tau}^\lambda), \beta_j, \delta^\lambda)\big). \tag{7.23}$$

Hence, for any $s_{-i} \in S_{-i}$ such that $\sigma_i(\{s_{-i}\} \mid \hat{\tau}^\lambda) > 0$ and any $s_j \in \langle s_{-i}\rangle$,

$$V_j\big(s_j \mid \alpha_j, \gamma_j(\sigma_j(\widehat{\tau}^\lambda), \beta_j, \delta^\lambda)\big) \geq V_j\big(\tilde{s}_j \mid \alpha_j, \gamma_j(\sigma_j(\widehat{\tau}^\lambda), \beta_j, \delta^\lambda)\big) \tag{7.24}$$

for all $\tilde{s}_j \in S_j$.

Suppose $\sigma(\{s_{-i}\} \mid \hat{\tau}^\kappa) > 0$ for some $\kappa \in \mathbb{N}$ and there is $\bar{s}_j \in \langle s_{-i}\rangle$ such that

$$V_j\big(\bar{s}_j \mid \alpha_j, \gamma_j(\sigma_j(\widehat{\tau}^\kappa), \beta_j, \delta^\kappa)\big) < V_j\big(\tilde{s}_j \mid \alpha_j, \gamma_j(\sigma_j(\widehat{\tau}^\kappa), \beta_j, \delta^\kappa)\big) \tag{7.25}$$

for some $\tilde{s}_j \in S_j$.

By continuity of V_j in δ and τ, there must be some $\mathcal{K} \in \mathbb{N}$ such that

$$V_j\big(\bar{s}_j \mid \alpha_j, \gamma_j(\sigma_j(\widehat{\tau}^\kappa), \beta_j, \delta^\kappa)\big) < V_j\big(\tilde{s}_j \mid \alpha_j, \gamma_j(\sigma_j(\widehat{\tau}^\kappa), \beta_j, \delta^\kappa)\big) \tag{7.26}$$

for all $\kappa > \mathcal{K}$, contradicting the premise that for any s_{-i} such that $\sigma_i(\{s_{-i}\} \mid \hat{\tau}^\lambda) > 0$ and any $s_j \in \langle s_{-i}\rangle$, we have that

$$V_j\big(s_j \mid \alpha_j, \gamma_j(\sigma_j(\widehat{\tau}^\lambda), \beta_j, \delta^\lambda)\big) \geq V_j\big(\tilde{s}_j \mid \alpha_j, \gamma_j(\sigma_j(\widehat{\tau}^\lambda), \beta_j, \delta^\lambda)\big) \tag{7.27}$$

□

7.4 Proofs of Propositions 4.1 and 4.2

Consider the game with strategy sets $S_i = T$ for every player $i \in I$. Recall from (4.1) the payoff function $u_i(s_i, s_{-i}) = A \min\{s_i, \underline{m}(s_{-i})\} - cs_i$, where $\underline{m}(s_{-i}) := \min\{s_j \in \langle s_{-i}\rangle : j \neq i\}$ denotes the lowest contribution of the opponents in the strategy combination s_{-i}.

For every contribution level $y \in T$, we denote by $s^y := (y, y, \dots, y) \in S = T^n$ the strategy combination where every player i chooses $s_i = y$, by s_{-i}^x the strategy combination where every player $j \neq i$ chooses $s_j = x$, and by $s_i^z = z$ the strategy of player i to contribute $z \in T$. Notice that $\underline{m}(s_{-i}^x) = x$ for all $x \in T$. Hence, the payoff of player

i from choosing strategy $s_i^y = y$ given that the opponents play s_{-i}^x can be written as a function $v(y,x)$ of $y, x \in T$:

$$v(y,x) := u(s_i^y, s_{-i}^x) = A \min\{y, x\} - cy. \tag{7.28}$$

Note also that the strategy combination s^y is a pure strategy, symmetric Nash equilibrium. The set of symmetric Nash equilibria is denoted by $\mathcal{N}(T) = \{s^y \in S \mid y \in T\}$.

Players know from the context of the game that contribution levels range between a minimum of m and a maximum of M. Let $C := \{y \in T | \ m \leq y \leq M\}$ be the set of contribution levels consistent with this context information and denote by $\mathcal{C} = \{s^y \in S \mid y \in C\}$ the set of all strategy profiles consistent with the context information C. Notice that $\mathcal{C} \subseteq \mathcal{N}(T)$.

We study symmetric equilibria in a symmetric game. Hence, given the symmetry of the players with respect to their degrees of ambiguity attitude, their degrees of confidence and their context information C, we drop the index i for α, δ and β. Denote by $\mathcal{C}_{-i} = \{s_{-i}^x \in S_{-i} \mid x \in C\}$ the set of the opponents' symmetric strategy combinations consistent with C. The belief that only strategies in C will be chosen is reflected by the mass distribution β on Σ_{-i} with $\beta(\mathcal{C}_{-i}) = 1$ and $\beta(E) = 0$ otherwise.

Consider an endogenous belief that the opponents contribute $x \in T$ represented by the probability distribution σ_i^x on S_{-i} concentrated on s_{-i}^x, that is $\sigma_i^x(\{s_{-i}^x\}) = 1$, then, by Proposition 2.1 and Equation (2.12), we obtain the CEU value $V_i\big(s_i^y | \alpha, \gamma(\sigma_i^x, \beta, \delta)\big)$ for a contribution $y \in T$ and an endogenous belief that the opponents contribute $x \in T$:

$$\begin{aligned}
V_i\big(s_i^y|\alpha, \gamma(\sigma_i^x, \beta, \delta)\big) &= \sum_{E \subseteq S_{-i}} \gamma_i(E \mid \sigma_i^x, \beta, \delta) V_i^\alpha(s_i^y, E) \\
&= (1-\delta) \sum_{s_{-i} \in S_{-i}} u(s_i^y, s_{-i}) \sigma_i^x(\{s_{-i}\}) + \delta \beta(\mathcal{C}_{-i}) V_i^\alpha(s_i^y, \mathcal{C}_{-i}) \\
&= (1-\delta) u(s_i^y, s_{-i}^x) + \delta V_i^\alpha(s_i^y, \mathcal{C}_{-i}) \\
&= (1-\delta) u(s_i^y, s_{-i}^x) + \delta \left[\alpha \min_{s_{-i} \in \mathcal{C}_{-i}} u\left(s_i^y, s_{-i}\right) + (1-\alpha) \max_{s_{-i} \in \mathcal{C}_{-i}} u\left(s_i^y, s_{-i}\right)\right] \\
&= (1-\delta) v(y,x) + \delta \left[\alpha v\left(y, m\right) + (1-\alpha) v\left(y, M\right)\right] \\
&= (1-\delta) A \min\{y, x\} + \delta \left[\alpha A \min\{y, m\} + (1-\alpha) A \min\{y, M\} - cy\right] \qquad (7.29) \\
&=: V(y|x)
\end{aligned}$$

For given α, δ and C (and, hence, β), we will write, for notational convenience, $V(y,x) := V_i\big(s_i^y | \alpha, \gamma(\sigma_i^x, \beta, \delta)\big)$.

We will denote the set of (pure-strategy) symmetric CD-EUAs associated with context information C by $\mathcal{M}(C, \alpha, \delta)$ for $\alpha \in [0,1]$ and $\delta \in [0,1]$, suppressing β in our notation and refering to C , the set of contributions defining the only set of strategy combinations $\mathcal{C}_{-i}$ with positive weight in the belief function β.[46] That is, $s^y = (s_i^y, s_{-i}^y) \in \mathcal{M}(C, \alpha, \delta)$

[46]We will suppress β in our notation throughout this section and refer, instead, to C, the set of contributions defining the strategy combinations with positive weight in the belief function β.

if and only if there is $\hat{\tau}$ such that, for each player i, $\text{supp}(\sigma_i(\hat{\tau})) = \{s^y_{-i}\}$ and $s^y_i = y \in R(\sigma(\hat{\tau}), C, \alpha, \delta)$, where $R(\sigma(\hat{\tau}), C, \alpha, \delta)$ is the best-reply correspondene (3.1) given the (equilibrium) belief function $\phi^{\gamma(\sigma(\hat{\tau}),\beta,\delta)}$ with β concentrated on on $\mathcal{C}_{-i}$. If there is no risk of misunderstanding. we will write $R(\alpha, \delta)$ instead of $R(\sigma(\hat{\tau}), C, \alpha, \delta)$.

In this environment, we explore the relationship between context information C and equilibrium behavior in $\mathcal{M}(C, \alpha, \delta)$. For each $y \in T$, we will determine the values of α and δ for which s^y is a pure-strategy CD-EUA, i.e., $s^y \in \mathcal{M}(C, \alpha, \delta)$. That is, we will determine the range of α and δ for which $V(y|y) \geq V(x|y)$ for all $x \in T$. If this is true, then $s^y \in \mathcal{M}(C, \alpha, \delta)$. First, we will prove the following auxilary lemma.

Lemma 7.1. *Let $\mathcal{N}(T)$ be the set of symmetric Nash-equilibria for a minimum-effort game and $\mathcal{M}(C, \alpha, \delta)$ the set of symmetric CD-EUAs associated with $\alpha, \delta \in [0, 1]$ and context information C. Then, for each strategy profile $s^y \in \mathcal{N}(T)$:*

(i) if $y < m$, then $s^y \in \mathcal{M}(C, \alpha, \delta)$ for any $\alpha \in [0, 1]$ and $\delta \in [0, \frac{c}{A})$;

(ii) if $y > M$, then $s^y \in \mathcal{M}(C, \alpha, \delta)$ for any $\alpha \in [0, 1]$ and $\delta \in [0, 1 - \frac{c}{A})$;

(iii) if $y = m$, then $s^y \in \mathcal{M}(C, \alpha, \delta)$ for any $\alpha \in [0, 1]$ and $\delta \in [0, \bar{\delta}(\alpha))$ where

$$\bar{\delta}(\alpha) := \min\left\{\frac{1}{(1-\alpha)}\frac{c}{A}, 1\right\};$$

(iv) if $y = M$, then $s^y \in \mathcal{M}(C, \alpha, \delta)$ for any $\alpha \in [0, 1]$ and $\delta \in [0, \bar{\delta}(\alpha))$ where

$$\bar{\delta}(\alpha) := \min\left\{\frac{1}{\alpha}\left(1 - \frac{c}{A}\right), 1\right\};$$

(v) if $m < y < M$, then $s^y \in \mathcal{M}(C, \alpha, \delta)$ for any $\alpha \in [0, 1]$ and $\delta \in \big[0, \bar{\delta}(\alpha)\big)$ where

$$\bar{\delta}(\alpha) := \min\left\{\frac{1}{\alpha}\left(1 - \frac{c}{A}\right), \frac{1}{(1-\alpha)}\frac{c}{A}\right\}.$$

Proof. The proof consists of five steps proving (i) through (v).

Step (i). Consider $s^y \in \mathcal{N}(T)$ such that $y < m$. Assume $\sigma(\{s^y_{-i}\}|\hat{\tau}) = 1$.

By Equation (7.29), the Choquet expected payoff for s^y_i w.r.t. $\sigma(\{s^y_{-i}\}|\hat{\tau}) = 1$ is

$$V(y \mid y) = \big(A - c\big)y. \tag{7.30}$$

Case 1. Consider $x \in T$ such that $x < y$. By (7.29), we have

$$V(x \mid y) = \big(A - c\big)x. \tag{7.31}$$

Since $(A - c) > 0$, by (7.30) and (7.31), we have $V(y \mid y) > V(x \mid y)$ for any $y > x$.

Case 2. Consider $x \in T$ such that $x > y$. Depending on the value of x, by (7.29), the Choquet expected payoff for x w.r.t. $\sigma(\{s_{-i}^y\}|\hat{\tau}) = 1$ is

$$V(x \mid y) = \begin{cases} (1-\delta)Ay + \delta Ax - cx, & \text{if } x \leq m, \\ (1-\delta)Ay + \delta A\big[\alpha m + (1-\alpha)x\big] - cx, & \text{if } m < x \leq M, \\ (1-\delta)Ay + \delta A\big[\alpha m + (1-\alpha)M\big] - cx, & \text{if } M < x. \end{cases} \tag{7.32}$$

For any $x \in T$ such that $y < x \leq m$, by (7.30) and (7.32), we have

$$\begin{aligned} V(y \mid y) &> V(x \mid y) \\ \big(A - c\big)y &> (1-\delta)Ay + \delta Ay - cy \\ c(y - y) &> \delta A\,(x - y) \\ \frac{c}{A} &> \delta. \end{aligned} \tag{7.33}$$

For any $x \in T$ such that $m < x \leq M$, by (7.32), when we compare the Choquet expected payoffs for m and x w.r.t. $\sigma(\{s_{-i}^y\}|\hat{\tau}) = 1$, we have

$$\begin{aligned} V(m \mid y) &> V(x \mid y) \\ (1-\delta)Ay + \delta Am - cm &> (1-\delta)Ay + \delta A\big[\alpha m + (1-\alpha)x\big] - cx \\ c(x - m) &> \delta A(1-\alpha)(x - m) \\ \frac{1}{(1-\alpha)}\frac{c}{A} &> \delta. \end{aligned} \tag{7.34}$$

Notice that $\frac{1}{(1-\alpha)}\frac{c}{A} \geq \frac{c}{A}$ if an only if $\alpha \geq 0$. Hence, (7.33) implies (7.34).

Finally, consider $x \in T$ such that $M < x$. By (7.32), since $c > 0$, we get

$$\begin{aligned} V(x \mid y) &> V(M \mid y) \\ -cM &> -cx \\ x &> M. \end{aligned} \tag{7.35}$$

Hence, for any $x > y$, we have $V(y \mid y) > V(x \mid y)$ for any $\alpha \geq 0$ and $\delta \in [0, \frac{c}{A})$.

Cases 1 and 2 show that $\{s_i^y\} = R(\alpha, \delta)$; i.e., s_i^y is the unique best reply with respect to $\gamma(\sigma(\hat{\tau}), \beta, \delta)$ based on $\sigma(\{s_{-i}^y\}|\hat{\tau}) = 1$ and $\beta(\mathcal{C}_{-i}) = 1$ for any $\alpha \in [0, 1]$ and $\delta \in [0, \frac{c}{A})$. For these parameters, we thus have that $s^y \in \mathcal{M}(C, \alpha, \delta)$ for any $y < m$.

Step (ii). Consider $s^y \in \mathcal{N}(T)$ such that $y > M$. Assume $\sigma_i(\{s_{-i}^y\}|\hat{\tau}) = 1$.

By Equation (7.29), the Choquet expected payoff for s_i^y w.r.t. $\sigma_i(\{s_{-i}^y\}|\hat{\tau}) = 1$ is

$$V(y \mid y) = (1-\delta)Ay + \delta A\big[\alpha m + (1-\alpha)M\big] - cy. \tag{7.36}$$

Case 1. For any $x > y$, by (7.29), we have

$$V(x \mid y) = (1-\delta)Ay + \delta A\big[\alpha m + (1-\alpha)M\big] - cx. \tag{7.37}$$

By comparing (7.36) and (7.37), since $c > 0$, we have

$$
\begin{aligned}
V(y \mid y) &> V(x \mid y) \\
-cy &> -cx \\
x &> y.
\end{aligned}
\tag{7.38}
$$

Case 2. Consider $x < y$. Depending on the value of x, by (7.29), we have

$$
V(x \mid y) = \begin{cases} (1-\delta)Ax + \delta A\big[\alpha m + (1-\alpha)M\big] - cx_i & \text{if } M \leq x, \\ (1-\delta\alpha)Ax + \delta\alpha Am - cx & \text{if } m < x < M, \\ \big(A - c\big)x & \text{if } x \leq m. \end{cases}
\tag{7.39}
$$

First, take $x \in T$ such that $M \leq x < y$. By comparing (7.36) and (7.39), we get

$$
\begin{aligned}
V(y \mid y) &> V(x \mid y) \\
(1-\delta)Ay + \delta A\big[\alpha y + (1-\alpha)M\big] - cy &> (1-\delta)Ax + \delta A\big[\alpha m + (1-\alpha)M\big] - cx \\
(1-\delta)A\,(y - x) &> c\,(y - x) \\
1 - \frac{c}{A} &> \delta.
\end{aligned}
\tag{7.40}
$$

Second, consider $x \in T$ such that $m < x < M$. By comparing the Choquet expected payoffs for M with that for x w.r.t. $\sigma_i(\{s^y_{-i}\}|\hat{\tau}) = 1$, we have

$$
\begin{aligned}
V(M \mid y) &> V(x \mid y) \\
(1-\delta\alpha)AM + \delta A\alpha m - cM &> (1-\delta\alpha)Ax + \delta\alpha Am - cx \\
(1-\delta\alpha)A\,(M - x) &> c\,(M - x) \\
\frac{1}{\alpha}\left(1 - \frac{c}{A}\right) &> \delta.
\end{aligned}
\tag{7.41}
$$

Notice that $\frac{1}{\alpha}\left(1 - \frac{c}{A}\right) \geq 1 - \frac{c}{A}$ if and only if $1 \geq \alpha$. Hence, (7.40) implies (7.41).

Finally, consider $x \in T$ such that $x \leq m$. The Choquet payoffs for m and x yield

$$
\begin{aligned}
V(m \mid y) &> V(x \mid y) \\
(A - c)\,m &> (A - c)\,x \\
m &> x.
\end{aligned}
\tag{7.42}
$$

Hence, for any $x < y$, $x \notin R(\alpha, \delta)$ for any $\alpha \in [0, 1]$ and $\delta \in [0, 1 - \frac{c}{A})$.

Cases 1 and 2 show that $\{s^y_i\} = R(\alpha, \delta)$; that is, y is the unique best reply to y with respect to $\gamma(\sigma(\hat{\tau}), \beta, \delta)$ based on $\sigma_i(\{s^y_{-i}\}|\hat{\tau}) = 1$ and $\beta(\mathcal{C}_{-i}) = 1$ for any $\alpha \in [0, 1]$ and $\delta \in [0, 1 - \frac{c}{A})$. For these parameters, we thus have that $s^y \in \mathcal{M}(C, \alpha, \delta)$.

Step (iii). Consider $s^m \in \mathcal{N}(T)$. Assume $\sigma_i(\{s^m_{-i}\}|\hat{\tau}) = 1$.

By Equation (7.29), the Choquet expected payoff for m w.r.r. $\sigma_i(\{s^m_{-i}\}|\hat{\tau}) = 1$ is

$$
V(m \mid m) = (1-\delta)Am + \delta Am - cm = \big(A - c\big)m.
\tag{7.43}
$$

Case 1. Consider $y \in T$ such that $y < m$. By Equation (7.29),

$$V(y \mid m) = (1-\delta)Ay + \delta Ay - cy = \big(A - c\big)y. \tag{7.44}$$

Since $(A - c) > 0$, by (7.43) and (7.44), $V(m \mid m) > V(y \mid m)$ for any $y < m$.

Case 2. Consider $y \in T$ such that $y > m$. Depending on y, by (7.29), we have

$$V(y \mid m) = \begin{cases} (1-\delta)Am + \delta A\big[\alpha m + (1-\alpha)y\big] - cy & \text{if } y \leq M, \\ (1-\delta)Am + \delta A\big[\alpha m + (1-\alpha)M\big] - cy & \text{if } y > M. \end{cases} \tag{7.45}$$

Hence, for any y such that $m < y \leq M$, by (7.43) and (7.45), we have

$$\begin{aligned} V(m \mid m) &> V(y \mid m) \\ \big(A - c\big)m &> (1-\delta)Am + \delta A\big[\alpha m + (1-\alpha)y\big] - cy \\ c\,(y - m) &> \delta\,(1-\alpha)\,A\,(y - m) \\ \frac{1}{(1-\alpha)}\frac{c}{A} &> \delta. \end{aligned} \tag{7.46}$$

Notice that $\frac{1}{(1-\alpha)}\frac{c}{A} \geq 1$ if and only if $\alpha \geq (1 - \frac{c}{A})$. Hence, (7.46) is satisfied for any $\delta \in [0, \bar{\delta}(\alpha))$ where $\bar{\delta}(\alpha) := \{\frac{1}{(1-\alpha)}\frac{c}{A}, 1\}$.

By (7.45), the Choquet payoffs for M and any $y > M$ w.r.r. $\sigma_i(\{s^m_{-i}\}|\hat{\tau}) = 1$ yield

$$\begin{aligned} V(M \mid m) &> V(y \mid m) \\ -cM &> -cy \\ y &> M. \end{aligned} \tag{7.47}$$

Hence, for any $y \in T$ such that $y > m$, $V(m \mid m) > V(y \mid m)$, showing that $y \notin R(\alpha, \delta)$ for any $\alpha \in [0, 1]$ and $\delta \in [0, \bar{\delta}(\alpha))$ where $\bar{\delta}(\alpha) := \{\frac{1}{(1-\alpha)}\frac{c}{A}, 1\}$.

Cases 1 and 2 show that $\{s^m_i\} = R(\alpha, \delta)$; i.e., m is the unique best reply with respect to $\gamma(\sigma(\hat{\tau}), \beta, \delta)$ based on $\sigma_i(\{s^m_{-i}\}|\hat{\tau}) = 1$ and $\beta(\mathcal{C}_{-i}) = 1$ for any $\alpha \in [0, 1]$ and $\delta \in [0, \bar{\delta}(\alpha))$ where $\bar{\delta}(\alpha) := \{\frac{1}{(1-\alpha)}\frac{c}{A}, 1\}$. For these parameters, we therefore have $s^m \in \mathcal{M}(C, \alpha, \delta)$.

Step (iv). Consider $s^M \in \mathcal{N}(T)$. Assume $\sigma_i(\{s^M_{-i}\}|\hat{\tau}) = 1$.

By (7.29), the Choquet expected payoff for M w.r.t. $\sigma_i(\{s^M_{-i}\}|\hat{\tau}) = 1$ is

$$V(M \mid M) = (1 - \delta\alpha)AM + \delta\alpha Am - cM_i. \tag{7.48}$$

Case 1. Consider $y > M$. By (7.29), the Choquet payoff for y w.r.t. $\sigma_i(s^M_{-i}\}|\hat{\tau}) = 1$ is

$$\begin{aligned} V(y \mid M) &= (1-\delta)AM + \delta A\big[\alpha m + (1-\alpha)M\big] - cy \\ &= (1 - \delta\alpha)AM + \delta\alpha Am - cy. \end{aligned} \tag{7.49}$$

Hence, by (7.48) and (7.49), since $c > 0$, we have

$$\begin{aligned} V(M \mid M) &> V(y \mid M) \\ -cM &> -cy \\ y &< M, \end{aligned} \tag{7.50}$$

showing that any $y > M$ cannot be a best reply, i.e., $y \notin R(\alpha, \delta)$.

Case 2. Consider $y < M$. Depending on the value of y, by (7.29), we have

$$V(y \mid M) = \begin{cases} (1-\delta\alpha)Ay + \delta\alpha Am - cy & \text{if } y > m, \\ (A-c)y & \text{if } y \leq m. \end{cases} \tag{7.51}$$

Hence, for any $y \in T$ such that $y < M$, by (7.48) and (7.51), we have

$$\begin{aligned} V(M \mid M) &> V(y \mid M) \\ (1-\delta\alpha)AM + \delta\alpha Am - cM &> (1-\delta\alpha)Ay + \delta\alpha Am - cy \\ (1-\delta\alpha)\,A\,(M-y) &> c\,(M-y) \\ (1-\delta\alpha) &> \frac{c}{A} \\ \frac{1}{\alpha}\left(1 - \frac{c}{A}\right) &> \delta. \end{aligned} \tag{7.52}$$

Notice that $\frac{1}{\alpha}\left(1 - \frac{c}{A}\right) \geq 1$ if and only if $\alpha \leq (1 - \frac{c}{A})$. Hence, (7.52) is satisfied for any $\delta \in [0, \bar{\delta}(\alpha))$ where $\bar{\delta}(\alpha) := \{\frac{1}{\alpha}\left(1 - \frac{c}{A}\right), 1\}$, showing that for any $y < M$, $s_i^M \notin R(\alpha, \delta)$.

Cases 1 and 2 show that $\{s_i^M\} = R(\alpha, \delta)$; i.e., M is the unique best reply with respect to $\gamma(\sigma(\hat{\tau}), \beta, \delta)$ based on $\sigma_i(\{M\}|\hat{\tau}) = 1$ and $\beta(\mathcal{C}_{-i}) = 1$ for any $\alpha \in [0, 1]$ and $\delta \in [0, \bar{\delta}(\alpha))$ where $\bar{\delta}(\alpha) := \{\frac{1}{(1-\alpha)} \frac{c}{A}, 1\}$. For these parameters, we therefore have $s^M \in \mathcal{M}(C, \alpha, \delta)$.

Step (v). Consider $s^y \in \mathcal{N}(T)$ where $m < y < M$. Assume $\sigma_i(\{s_{-i}^y\}|\hat{\tau}) = 1$.

By (7.29), the Choquet expected payoff for y w.r.t. $\sigma_i(\{s_{-i}^y\}|\hat{\tau}) = 1$ is

$$\begin{aligned} V(y \mid y) &= (1-\delta)Ay + \delta A\big[\alpha m + (1-\alpha)y\big] - cy \\ &= (1-\delta\alpha)Ay + \delta\alpha Am - cy. \end{aligned} \tag{7.53}$$

We will consider three cases.

Case 1. Consider $x < m$. By comparing the Choquet expected payoffs for m and that for $x < m$ w.r.t. $\sigma_i(\{s_{-i}^y\}|\hat{\tau}) = 1$, since $(A - c) > 0$, we have

$$\begin{aligned} V(m \mid y) &> V(x \mid y) \\ \big(A-c\big)m &> \big(A-c\big)x \\ m &> y. \end{aligned} \tag{7.54}$$

Case 2. Consider $x > M$. By comparing the Choquet expected payoffs for M and that for any $x > M$ w.r.t. $\sigma_i(\{s_{-i}^y\}|\hat{\tau}) = 1$, since $c > 0$, we have

$$\begin{aligned} V(M \mid y) &> V(x \mid y) \\ -cM &> -cy \\ y &> M. \end{aligned} \tag{7.55}$$

Case 3. Consider now $x \in T$ such that $m \leq x \leq M$ and $x \neq y$.

First, consider $x \in T$ such that $y < x \leq M$. By (7.29), the Choquet expected payoff for x w.r.t. $\sigma_i(\{s_{-i}^y\}|\hat{\tau}) = 1$ is

$$V(x \mid y) = (1-\delta)Ay + \delta A\big[\alpha m + (1-\alpha)x\big] - cx. \tag{7.56}$$

By comparing the Choquet payoffs for y with that for x, by (7.54), (7.56), we have

$$\begin{aligned} V(y \mid y) &> V(x \mid y) \\ (1-\delta)Ay + \delta A\big[\alpha m + (1-\alpha)y\big] - cy &> (1-\delta)Ay + \delta A\big[\alpha m + (1-\alpha)x\big] - cx \\ \delta A(1-\alpha)y - cy &> \delta A(1-\alpha)x - cx \\ c\,(x-y) &> (1-\alpha)\delta A\,(x-y) \\ \frac{1}{(1-\alpha)} &> \delta \end{aligned} \tag{7.57}$$

Notice that when $\alpha \geq 1 - \frac{c}{a}$, the above inequality is satisfied for any $\delta < 1$.

Now, consider $x \in T$ such that $m \leq x < y$. By (7.29), we have

$$\begin{aligned} V(x \mid y) &= (1-\delta)Ax + \delta A\big[\alpha m + (1-\alpha)x\big] - cx \\ &= (1-\delta\alpha)Ax + \delta\alpha Am - cx. \end{aligned} \tag{7.58}$$

By comparing (7.53) and (7.58), we have

$$\begin{aligned} V(y \mid y) &> V(x \mid y) \\ (1-\delta\alpha)Ay + \delta\alpha Am - cy &> (1-\delta\alpha)Ax + \delta\alpha Am - cx \\ (1-\delta\alpha)\,A\,(y-x) &> c\,(y-x) \\ \left(1 - \frac{c}{A}\right) &> \delta\alpha \\ \frac{1}{\alpha}\left(1 - \frac{c}{A}\right) &> \delta. \end{aligned} \tag{7.59}$$

Notice that when $\alpha \geq 1 - \frac{c}{A}$, the above inequality is satisfied for any $\delta < 1$.

For y to be optimal, δ must take values for which (7.57) and (7.59) are simultaneously satisfied. On the one hand, when $\alpha > 1 - \frac{c}{A}$, we have $\frac{1}{(1-\alpha)}\frac{c}{A} > 1 > \frac{1}{\alpha}(1 - \frac{c}{A})$. In this case, (7.57) is satisfied for any $\delta < 1$ while (7.59) is satisfies only for $\delta < \frac{1}{\alpha}\left(1 - \frac{c}{a}\right)$. On the other hand, when $\alpha < 1 - \frac{c}{A}$, we have $\frac{1}{(1-\alpha)}\frac{c}{A} < 1 < \frac{1}{\alpha}(1 - \frac{c}{A})$. In this case, (7.57) is satisfied for any $\delta < \frac{1}{(1-\alpha)}\frac{c}{A}$ while (7.59) is satisfies for any $\delta < 1$. Moreover, when $\alpha = 1 - \frac{c}{A}$, (7.57) and (7.59) are satisfied for any $\delta < 1$ since $\frac{1}{(1-\alpha)}\frac{c}{A} = \frac{1}{\alpha}(1 - \frac{c}{A}) = 1$.

Summing up Case 3, for any $x \in T$ such that $m \leq x \leq M$ and $x \neq y$, we have shown that $V(y \mid y) > V(x \mid y)$ for any $\alpha \in [0,1]$ and $\delta \leq \bar{\delta}(\alpha)$ where

$$\bar{\delta}(\alpha) := \min\left\{\frac{1}{\alpha}\left(1-\frac{c}{A}\right), \frac{1}{(1-\alpha)}\frac{c}{A}\right\}. \tag{7.60}$$

From Cases 1, 2, 3, we thus conclude that for any $y \in T$ such that $m < y < M$, we have $\{s_i^y\} = R(\alpha, \delta)$. That is, $s_i^y = y$ is the unique best reply to s_{-i}^y with respect to $\gamma(\sigma(\hat{\tau}), \beta, \delta)$ based on $\sigma_i(\{s_{-i}^y\}|\hat{\tau}) = 1$ and $\beta(\mathcal{C}_{-i}) = 1$ for any $\alpha \in [0,1]$ and $\delta \in [0, \bar{\delta}(\alpha))$. For these parameters, we get $s^y \in \mathcal{M}(C, \alpha, \delta)$ for any y such that $m < y < M$. Steps (i) through (v) complete the proof of Lemma 7.1 □

Proof of Proposition 4.1. Let $\mathcal{C} = \{s^y \in S \mid y \in C\}$ be the set of strategy profiles consistent with the context information C.

Case 1. Notice that $\frac{1}{\alpha}\left(1-\frac{c}{A}\right) \geq 1$ if and only if $\alpha \geq 1 - \frac{c}{A}$. By Lemma 7.1 (*iii*), for any $\alpha \in \left(1-\frac{c}{A}, 1\right]$ and thus any $\delta \in [0,1)$ the strategy profile $s^m \in \mathcal{C}$ is a pure CD-EUA. That is, $s^m \in \mathcal{M}(C, \alpha, \delta)$ and thus $\mathcal{M}(C, \alpha, \delta) \cap \mathcal{C} \neq \emptyset$.

Case 2. Notice that $\frac{1}{(1-\alpha)}\frac{c}{A} \geq 1$ if and only if $\alpha \leq 1 - \frac{c}{A}$. By Lemma 7.1 (*iv*), for any $\alpha \in [0, 1-\frac{c}{A})$ and thus any $\delta \in [0,1)$, the strategy profile $s^M \in \mathcal{C}$ is a pure CD-EUA. That is, $s^M \in \mathcal{M}(C, \alpha, \delta)$ and thus $\mathcal{M}(C, \alpha, \delta) \cap \mathcal{C} \neq \emptyset$.

Case 3. Notice that $\frac{1}{(1-\alpha)}\frac{c}{A} = \frac{1}{\alpha}\left(1-\frac{c}{A}\right)$ if and only if $\alpha = 1 - \frac{c}{A}$. By Lemma 7.1 (*v*), if $\alpha = 1 - \frac{c}{A}$ then for any $\delta \in [0,1)$ and any $y \in (m, M)$, $s^y \in \mathcal{C}$ is a pure CD-EUA. That is, $s^y \in \mathcal{M}(C, \alpha, \delta)$ and thus $\mathcal{M}(C, \alpha, \delta) \cap \mathcal{C} \neq \emptyset$.

Case 4. Consider now the case where $\delta = 1$; i.e., players are fully confident in the context information C. In this case, the equilibrium behavior depends solely α and not on the endogenous belief. In particular, $\{s^m\} = \mathcal{M}(C, \alpha, 1)$ for any $\alpha \in (1-\frac{c}{A}, 1]$; $\{s^M\} = \mathcal{M}(C, \alpha, 1)$ for any $\alpha \in [0, 1-\frac{c}{A})$; and for $m \leq y \leq M$, $s^y \in \mathcal{M}(C, \alpha, 1)$ for $\alpha = 1 - \frac{c}{A}$. Hence, for $\delta = 1$, $\mathcal{M}(C, \alpha, \delta) \cap \mathcal{C} \neq \emptyset$ for any $\alpha \in [0,1]$.

Cases 1, 2, 3 and 4 show that for any $\delta \in [0,1]$ and $\alpha \in [0,1]$, $\mathcal{M}(C, \alpha, \delta) \cap \mathcal{C} \neq \emptyset$. □

Proof of Proposition 4.2. Let $\mathcal{C} = \{s^y \in S \mid y \in C\}$ be the set of strategy profiles consistent with the context information C. Let $\bar{\delta} \in [0,1]$ be defined as follows:

$$\bar{\delta} := \max\{\frac{c}{A}, 1 - \frac{c}{A}\}. \tag{7.61}$$

By Lemma 7.1 (*i*) and (*ii*), we have that for any $\delta \geq \bar{\delta}$, the strategy profile $s^y \in \mathcal{C}$ such that $y < m$ or $y > M$ is not a (pure-strategy) CD-EUA, i.e., $s^y \notin \mathcal{M}(C, \alpha, \delta)$.

For any $\delta \in (\bar{\delta}, 1]$, $\{s^m\} = \mathcal{M}(C, \alpha, \delta)$ for any $\alpha \in \left(1-\frac{c}{A}, 1\right]$ by Lemma 7.1 (*iii*); $\{s^M\} = \mathcal{M}(C, \alpha, \delta)$ for any $\alpha \in \left[0, 1-\frac{c}{A}\right]$ by Lemma 7.1 (*iv*); and $s^y \in \mathcal{M}(C, \alpha, \delta)$ where $m < y < M$ for $\alpha = 1 - \frac{c}{A}$ by Lemma 7.1 (*v*).

Hence, for any $\alpha \in [0,1]$ and $\delta \geq \bar{\delta}$, $\mathcal{M}(C, \alpha, \delta) \subset \mathcal{C}$. □

We will prove Corollary 4.1 by proving a more general version.

Corollary 7.1. *Consider a minimum-effort game. Let $\mathcal{M}(C, \alpha, \delta)$ be the set of (pure-strategy) symmetric CD-EUAs for $\alpha, \delta \in [0,1]$ and context information C. Then,*

(*i*) *if* $\alpha > \frac{c}{A}$, *then* $\bigcap_{\delta \in [0,1]} \mathcal{M}(C, \alpha, \delta) := \{s^m\}$;

(*ii*) *if* $\alpha < \frac{c}{A}$, *then* $\bigcap_{\delta \in [0,1]} \mathcal{M}(C\,\alpha, \delta) := \{s^M\}$;

(*iii*) *if* $\alpha = \frac{c}{A}$, *then* $\bigcap_{\delta \in [0,1} \mathcal{M}(C, \alpha, \delta) := \{s^y \in S \mid m \leq y \leq M\}$.

Proof. Consider s^m. By Case 1 and 4 in the proof of Proposition 4.1, $s^m \in \mathcal{M}(C, \alpha, \delta)$ for any $\alpha \in \left(1 - \frac{c}{A}\right]$ and $\delta \in [0,1]$. By the proof of Proposition 4.2, $\{s^m\} = \mathcal{M}(C, \alpha, \delta)$ for any $\alpha \in \left(1 - \frac{c}{A}\right]$ and $\delta \in [\bar{\delta}, 1]$ with $\bar{\delta}$ as in (7.61). Hence, the intersection of all pure-strategy CD-EUAs over all degrees of confidence $\delta \in [0,1]$ is the single strategy profile s^m; that is,

$$\{s^m\} := \bigcap_{\delta \in [0,1]^n} \mathcal{M}(C, \alpha, \delta).$$

Cases (*ii*) and (*iii*) follow from the other cases in the proof of Proposition 4.1 and 4.2. □

Corollary 4.1 follows from Corollary 7.1 since $\frac{c}{A} = \frac{1}{2}$ for $A = 2c$.

7.5 Proof of Proposition 5.1

Let m be the minimum contribution according to the context information. Let $\underline{m}(s_{-i}) := \min\{s_j \in \langle s_{-i} \rangle\}$ be the smallest contribution associated with the strategy combination $s_{-i} \in S_{-i}$. Denote by $S_{-i}(m) := \{s_{-i} \in S_{-i} : m \leq \underline{m}(s_{-i})\}$ be the set of strategy combinations of the opponents which yield a minimal contribution level of at least m.

Proof. Let $(\alpha_1, \ldots, \alpha_n) \in [0,1)^n$ be a vector of degrees of ambiguity attitudes, $\beta(m) := (\beta_1(m), \ldots, \beta_n(m))$ be the exogenous context information and $\delta = (\delta_1, \ldots, \delta_n) \in [0,1]^n$ be a vector of confidence parameters. By Theorem 3.1 there are probability distributions $\hat{\tau} = (\hat{\tau}_1, \ldots, \hat{\tau}_n) \in \times_{i \in I} \Delta(S_i)$ for which the n-tuple of beliefs functions

$$\left(\phi_1^{\gamma_1(\sigma_1(\hat{\tau}), \beta_1(m), \delta_1)}, \ldots, \phi_n^{\gamma_n(\sigma_n(\hat{\tau}), \beta_n(m), \delta_n)}\right)$$

defined by the mass distributions (see Definitions 2.1 and 3.3)

$$\left(\gamma_1(\sigma_1(\hat{\tau}), \beta_1(m), \delta_1), \ldots, \gamma_n(\sigma_n(\hat{\tau}), \beta_n(m), \delta_n)\right)$$

is a CD-EUA.

Fix a player $i \in I$. We will show that for all $\sigma_i(\tau)$ and all $s_i \in S_i$ such that $s_i < m$

$$V_i(m \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i(m), \delta_i)) > V_i(s_i \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i(m), \delta_i)). \quad (7.62)$$

Hence, in particular, for the Choquet payoff with respect to $\sigma_i(\hat{\tau})$, we have

$$V_i(m \mid \alpha_i, \gamma_i(\sigma_i(\hat{\tau}), \beta_i(m), \delta_i)) > V_i(s_i \mid \alpha_i, \gamma_i(\sigma_i(\hat{\tau}), \beta_i(m), \delta_i)). \tag{7.63}$$

for all $s_i < m$. By Proposition 2.1, the Choquet expected payoff is given by

$$\begin{aligned}
&V_i(s_i \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i(m), \delta_i)) = \sum_{E \subseteq S_{-i}} \gamma_i(E \mid \sigma_i(\tau), \beta_i(m), \delta_i) \cdot V_i^{\alpha_i}(s_i, E) \\
&= (1-\delta_i) \sum_{s_{-i} \in S_{-i}} \sigma_i(\{s_{-i}\}|\tau) \cdot V_i^{\alpha_i}(s_i, s_{-i}) + \delta_i \beta_i(S_{-i}(m)) \cdot V_i^{\alpha_i}(s_i, S_{-i}(m)) \\
&= (1-\delta_i) \sum_{s_{-i} \in S_{-i}} \sigma_i(\{s_{-i}\}|\tau) \cdot u_i(s_i, s_{-i}) \\
&\quad + \delta_i \left[\alpha_i \min_{s_{-i} \in S_{-i}(m)} u_i(s_i, s_{-i}) + (1-\alpha_i) \max_{s_{-i} \in S_{-i}(m)} u_i(s_i, s_{-i})\right] \\
&= d + (1-\delta_i) \sum_{s_{-i} \in S_{-i}} \sigma_i(\{s_{-i}\}|\tau) \big[a \min\{s_i, \underline{m}(s_{-i}) \geq\} - cs_i\big] \\
&\quad + \delta_i \Big[\alpha_i \big(a \min\{s_i, m\} - cs_i\big) + (1-\alpha_i) a \big(\min\{s_i, s^{max}\} - cs_i\big)\Big] \\
&= d + (1-\delta_i) \sum_{s_{-i} \in S_{-i}} \sigma_i(\{s_{-i}\}|\tau) \big[a \min\{s_i, \underline{m}(s_{-i}) \geq\} - cs_i\big] \\
&\quad + \delta_i \Big[\alpha_i a \min\{s_i, m\} + (1-\alpha_i) a \min\{s_i, s^{max}\}\Big] - \delta_i c s_i
\end{aligned} \tag{7.64}$$

Since $m \leq s^{max}$, for any $s_i < m$, we have that $s_i = \min\{s_i, m\} = \min\{s_i, s^{max}\}$, whereas for $s_i = m$, we get $m = \min\{s_i, m\} = \min\{s_i, s^{max}\}$. Therefore,

$$V_i(s_i \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i(m), \delta_i)) = \tag{7.65}$$

$$= \begin{cases} d + (1-\delta_i) \sum_{s_{-i} \in S_{-i}} \sigma_i(\{s_{-i}\}|\tau) \big[a \min\{s_i, \underline{m}(s_{-i}) \geq\} - cs_i\big] + \delta_i (a-c) s_i & \text{for } s_i < m \\ d + (1-\delta_i) \sum_{s_{-i} \in S_{-i}} \sigma_i(\{s_{-i}\}|\tau) \big[a \min\{m, \underline{m}(s_{-i}) \geq\} - cm\big] + \delta_i (a-c) m, & \text{for } s_i = m \end{cases}$$

Case 1: Take $s_i < m$. By 7.65, we have

$$\begin{aligned}
&V_i(s_i \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i(m), \delta_i)) \\
&= d + (1-\delta_i) \sum_{s_{-i} \in S_{-i}} \sigma_i(\{s_{-i}\}|\tau) \big[a \min\{s_i, \underline{m}(s_{-i}) \geq\} - cs_i\big] + \delta_i (a-c) s_i \\
&\leq d + (1-\delta_i)(a-c) s_i + \delta_i (a-c) s_i = d + (a-c) s_i \\
&\leq d + (a-c)(m-1) =: A
\end{aligned}$$

Case 2: Take $s_i = m$. By 7.65, we have

$$
\begin{aligned}
& V_i(m \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i(m), \delta_i)) \\
&= d + (1-\delta_i) \sum_{s_{-i}\in S_{-i}} \sigma_i(\{s_{-i}\}|\tau)\big[a \min\{m, \underline{m}(s_{-i}) \geq\} - cm\big] + \delta_i\,(a-c)\,m \\
&\geq d + (1-\delta_i) \sum_{s_{-i}\in S_{-i}} \sigma_i(\{s_{-i}\}|\tau)\big[a \min\{m, s^{\min}\} - cm\big] + \delta_i\,(a-c)\,m \\
&= d + (1-\delta_i)\,\big[as^{\min} - cm\big] + \delta_i\,(a-c)\,m \\
&= d + \big(as^{\min} - cm\big) + \delta_i a\left(m - s^{\min}\right) =: B.
\end{aligned}
$$

From Cases 1 and 2, we have

$$
B = d + \big(as^{\min} - cm\big) + \delta_i a\left(m - s^{\min}\right) > d + (a-c)\,(m-1) = A \tag{7.66}
$$

if and only if

$$
\delta_i \geq \overline{\delta} := \frac{a\left(m - s^{\min}\right) - (a-c)}{a\left(m - s^{\min}\right)} \tag{7.67}
$$

Hence,

$$
V_i(m \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i(m), \delta_i)) \geq B > A \geq V_i(s_i \mid \alpha_i, \gamma_i(\sigma_i(\tau), \beta_i(m), \delta_i)). \tag{7.68}
$$

for all $s_i < m$, showing that $m \leq \underline{m}(s_{-i}) \geq$ for all $s_{-i} \in \operatorname{supp}\sigma_i(\hat{\tau})$, all players $i \in I$ and any vector of degrees of ambiguity attitude $(\alpha_1, \ldots, \alpha_n) \in [0,1]^n$ □

8 Capacities and Choquet integral

In this appendix, we provide a brief overview of the Choquet integral for capacities and belief functions. For notational convenience, we will drop the indexes referring to players.

Let S be a finite set of states, Σ the set of all subsets (events) of S.[47] For any $E \in \Sigma$, let E^c denote the complement of E. Let $X \subseteq \mathbb{R}$ be a set of outcomes.[48] An *act* is a function $f : S \to X$, and $\mathcal{F}$ denotes the set of such acts. Given any two acts f and g in $\mathcal{F}$ and E in Σ, we denote by $f_E g \in \mathcal{F}$ the act defined as $f_E g(s) = f(s)$ if $s \in E$ and $f_E g(s) = g(s)$ otherwise. We will not distinguish between the outcome $x \in X$ and the constant act $x \in \mathcal{F}$, defined as $x(s) = x$ for all $s \in S$.

[47] In game-theoretic applications, this is the set of strategy combinations of the opponents S_{-i}.

[48] For ease of exposition, we restrict here attention to real-valued outcomes. This is without loss of generality, because all axiomatic treatments derive a real-valued von Neumann Morgenstern utility function on arbitrary sets of outcomes.

8.1 Capacities

Definition 8.1. A *capacity* ν is a set function $\nu : \Sigma \to \mathbb{R}$ which satisfies

(*i*) normalisation: $\nu(\emptyset) = 0$ and $\nu(S) = 1$, and

(*ii*) monotonicity: $\nu(A) \leq \nu(B)$ for all $A, B \in \Sigma$ such that $A \subset B$

The *set of all capacities* on Σ will be denoted by $\mathcal{V}$.

A finite-outcome act is an act where $f(S) = \{x_1, x_2, \ldots, x_n\}$. Without loss of generality, one can assume that the finite outcomes $x_i \in f(S)$ are ordered such that $x_1 \leq \ldots \leq x_i \leq \ldots \leq x_n$. The Choquet integral of a finite-outcome act f with respect to the capacity ν is defined as

$$\int f d\nu := \sum_{i=1}^{n} x_i \cdot \left[\nu(A_i \cup A_{i+1} \cup \ldots \cup A_n) - \nu(A_{i+1} \cup A_{i+2} \cup \ldots \cup A_n)\right] \tag{8.1}$$

where $A_i = f^{-1}(x_i)$ is the set of states that yield the outcome x_i and $A_{n+1} = \varnothing$.

Equivalently, one can also write the Choquet integral of a capacity ν as

$$\int f d\nu := \sum_{x \in f(S)} x \cdot \Big[\nu\big(\{s \mid f(s) \geq x\}\big) - \nu\big(\{s \mid f(s) > x\}\big)\Big] \tag{8.2}$$

Proposition 8.1. *(Denneberg, 2000, p.49) For capacities $\nu, \mu, \mu' \in \mathcal{V}$ and any number $\alpha \in [0, 1]$,*

- $\nu(E) = \alpha\mu(E)$ *for all* $E \subseteq S$, $\alpha \geq 0 \implies \int f d\nu = \alpha \int f d\mu$
- $\nu(E) = \mu(E) + \mu'(E)$ *for all* $E \subseteq S$, $\implies \int f d\nu = \int f d\mu + \int f d\mu'$

8.2 Dual capacities

For any capacity $\nu \in \mathcal{V}$ there is an associate dual capacity $\bar{\nu}$ defined by $\bar{\nu}(E) := 1 - \nu(E^c)$ for all $E \in \Sigma$. By definition, $\nu(E) + \bar{\nu}(E^c) = 1$. Moreover, if ν is additive, i.e., if the capacity is a probability distribution, then $\nu(E) + \bar{\nu}(E) = 1$. For *convex* capacities (defined below), one has $\nu(E) + \bar{\nu}(E) \leq 1$.

The Choquet integral of an act with respect to the dual capacity $\bar{\mu}$ is

$$\begin{aligned}\int f d\overline{\nu} &= \sum_{i=1}^{n} x_i \cdot \left[\overline{\mu}(A_i \cup A_{i+1} \cup \ldots \cup A_n) - \overline{\mu}(A_{i+1} \cup A_{i+2} \cup \ldots \cup A_n)\right] \\ &= -\sum_{i=1}^{n} x_i \cdot \left[\mu(A_{i+1} \cup A_{i+2} \cup \ldots \cup A_n) - \mu(A_i \cup A_{i+1} \cup \ldots \cup A_n))\right] = -\int (-f) d\mu.\end{aligned}$$

8.3 Möbius transforms

The set of all capacities $\mathcal{V}$ is a vector space of dimension $|\Sigma|$. For each element $T \in \Sigma$, a capacity u_T (called *elementary belief function* or *unanimity game*) can be defined by

$$u_T(E) := \begin{cases} 1 & \text{for} \quad T \subseteq E, \\ 0 & \text{otherwise.} \end{cases} \tag{8.3}$$

Lemma 8.1. *(Gilboa and Schmeidler, 1994, Lemma 4.2) For all $f \in \mathcal{F}$ and all $T \in \Sigma$,*

$$\int f du_T = \min\{f(s) : s \in T\}. \tag{8.4}$$

Proposition 8.2. *(Gilboa and Schmeidler, 1994, Theorem 3.3) The family of elementary belief functions $\{u_T\}_{T\in\Sigma\setminus\varnothing}$ is a linear basis of V. The unique set of coefficients $\{\beta_\nu(T)\}_{T\in\Sigma\setminus\varnothing}$ satisfying*

$$\nu = \sum_{T\in\Sigma\setminus\varnothing} \beta_\nu(T) u_T \tag{8.5}$$

is given by

$$\beta_\nu(T) = \sum_{E\subseteq T} (-1)^{|T|-|E|} \nu(E) = \nu(T) - \sum_{\{I|\ \varnothing\neq I\subseteq\{1,2,\ldots,n\}\}} (-1)^{|I|+1} \nu(\bigcap_{i\in I} T_i)$$

where $T = \{s_1, \ldots, s_{|T|}\}$ and $T_i = T\setminus\{s_i\}$.

The vector of coefficients $\beta_\nu(T)$, $T \subseteq S$, is called *Möbius transform* of the capacity ν. The capacity ν can be derived from its Möbius transform β according to the formula

$$\nu(A) = \sum_{T\subseteq A} \beta(T) \quad \text{for all } A \in \Sigma. \tag{8.6}$$

9 Special types of capacities

9.1 Convex capacities

A capacity $\nu \in \mathcal{V}$ is *convex* (*concave*) if for all $A, B \in \Sigma$:

$$\nu(A \cup B) + \nu(A \cap B) \geq (\leq)\, \nu(A) + \nu(B). \tag{9.1}$$

Definition 9.1. The *core* of a capacity ν is the set of probability distributions on S which event-wise dominate ν:

$$\text{core}(\nu) := \big\{p \in \Delta(S) :\ p(E) \geq \nu(E),\ E \subseteq S\big\}, \tag{9.2}$$

where $\Delta(S)$ denotes the simplex of S and $p(E) := \sum_{s\in E} p(s)$ the probability of event E.

The following result is well-known.

Lemma 9.1. *(Shapley, 1971; Schmeidler, 1989) For a convex capacity* ν*,*

$$\begin{aligned} &(i) \quad \text{core}(\nu) \neq \varnothing, \\ &(ii) \quad \int f d\nu = \min_{p \in \text{core}(\nu)} \int f \ dp, \end{aligned}$$

where $\int f \ dp$ *denotes the Riemann integral of* f *with respect to the probability distribution* p*.*

The dual capacity of a convex capacity is a concave capacity: ν convex $\Longrightarrow \overline{\nu}$ concave.

Definition 9.2. The *anti-core* of capacity ν, $\overline{\text{core}}(\nu)$ is the set of probability distributions on S which are dominated by ν for all events,

$$\overline{\text{core}}(\nu) := \{p \in \Delta(S) : \ p(E) \leq \nu(E), \ E \subseteq S\}. \tag{9.3}$$

Lemma 9.2. *The core of a convex capacity* ν *equals the anti-core of the dual* $\overline{\nu}$*:*

$$\text{core}(\nu) = \overline{\text{core}}(\overline{\nu}).$$

Hence, Lemma B.2 implies that $\text{core}(\nu) \neq \varnothing$ iff $\overline{\text{core}}(\overline{\nu}) \neq \varnothing$. Moreover,

$$\int f d\overline{\nu} = \max_{p \in \overline{\text{core}}(\overline{\nu})} \int f \ dp = \min_{p \in \text{core}(\nu)} \int f \ dp, \tag{9.4}$$

since $p(E) \geq \nu(E) \iff (1 - p(E^c)) \geq \nu(E) = 1 - \overline{\nu}(E^c) \iff \overline{\nu}(E^c) \geq p(E^c)$ and $\overline{\nu}(E^c) + \nu(E) \leq 1$ for a convex capacity ν.

9.2 JP-capacities

For a convex capacity μ, one can interpret the capacity as a lower probability and its dual capacity as an upper probability, since $\mu(E) \leq p(E) = 1 - p(E^c) \leq \overline{\mu}(E)$ for all $E \in \Sigma$. Hence, Jaffray and Philippe (1997) suggest convex combinations of a convex capacity μ and its dual $\bar{\mu}$ as a capacity which is a weighted average of the lowest possible probability of an event and the highest probability of the event. The JP-capacity is given by

$$\nu^{JP}(E) := \alpha\mu(E) + (1 - \alpha)\overline{\mu}(E). \tag{9.5}$$

for each $E \in \Sigma$. In general, JP-capacities are neither concave nor convex.

Proposition B.1 yields immediately the following identity:

$$\int f d\nu^{JP} = \int f d(\alpha\mu + (1 - \alpha)\overline{\mu}) = \alpha \int f d\mu + (1 - \alpha) \int f d\overline{\mu}. \tag{9.6}$$

Moreover, by Lemma B.2 and B.3

$$\begin{aligned} \int f d\nu^{JP} &= \alpha \int f d\mu + (1 - \alpha) \int f d\overline{\mu} \\ &= \alpha \min_{p \in \text{core}(\mu)} \int f \ dp + (1 - \alpha) \max_{p \in \text{core}(\mu)} \int f \ dp. \end{aligned} \tag{9.7}$$

Hence, one can view the Choquet integral of a JP-capacity as an α-MEU representation with respect to the set of priors given by the core(μ). It is important to keep in mind that the capacity μ must be convex for a JP-capacity to be well defined. It is immediately clear that $\alpha = 1$ yields the Choquet integral w.r.t. the convex capacity μ and the case $\alpha = 0$ the Choquet integral w.r.t. the concave dual capacity $\overline{\mu}$.

9.3 Belief functions

For capacities, one can define different classes of monotonicity. The capacity itself is 1-monotone, a convex capacity is 2-monotone, etc. A capacity which is *totally monotone*, i.e., monotone up to any degree, is called a *belief function.*[49]

A capacity ν is a belief function if and only if its Möbius transform, i.e., the vector of coefficients $\beta_\nu(T)$, $T \subseteq S$, is non-negative and sums to 1, $\sum_{T\in\Sigma} \beta_\nu(T) = 1$. Belief functions are convex capacities.

A *mass distribution* m on Σ is a probability distribution on Σ, i.e., $m(E) \geq 0$ for all $E \in \Sigma$ and $\sum_{E\in\Sigma} m(E) = 1$.[50] Note that m is a set function but not a capacity, since it is not monotone and not normalized. There is, however, a unique belief function ν^m associated with m which is defined as $\nu^m(E) := \sum_{F\subseteq E} m(F)$ for all $E \in \Sigma$. It is easy to check that ν^m is monotone and normalized and, hence, a capacity with m as its Möbius transform.

Mass distributions can model elementary beliefs about the probability of events. Obviously, for any two mass distributions m, m' and any $\delta \in [0, 1]$, $\delta m + (1-\delta)m'$ is another mass distribution. Hence, given two belief functions ν and ν' and $\delta \in [0, 1]$, $\delta\nu + (1-\delta)\nu'$ is again a belief function.

Unanimity games u_T are (elementary) belief functions. Recall form Lemma B.1 that

$$\int f du_T = \min\{f(s) : s \in T\}$$

for all $f \in \mathcal{F}$ and all $T \in \Sigma$.

The Choquet integral of a belief function ν^m derived from a mass distribution m is the weighted sum over all events $T \in \Sigma$ of the minimum outcome on the events $\min\{f(s) : \ s \in T\}$ weighted by the mass distribution m.

Theorem 9.1. *(Gilboa and Schmeidler, 1994, Theorem 4.3) Let ν^m be a belief function. Then, for all $f \in \mathcal{F}$*

$$\int f d\nu^m = \sum_{T\in\Sigma\setminus\varnothing} m(T)\big[\min\{f(s) : \ s \in T\}\big]. \tag{9.8}$$

[49]For more details, see Gilboa and Schmeidler (1994, p.47) or Denneberg (2000, p.45).

[50]There is no uniform terminology in the literature. What we call *mass distribution* is sometimes called *belief function, mass function*, or *basic probability assignment* (see Grabisch, 2016, p.379-380).

Consider a JP-capacity with a belief function ν^m as its convex part. Let m be a mass distribution and $(\nu^m)^{JP}$ the JP-capacity based on the belief function ν^m, then

$$\int fd\left(\nu^m\right)^{JP} = \int fd\big(\alpha\nu^m + (1-\alpha)\overline{\nu^m}\big) = \alpha \int fd\nu^m + (1-\alpha)\int fd\overline{\nu^m}$$
$$= \sum_{E\in\Sigma} m(E)\left[\alpha \min_{s\in E} f(s) + (1-\alpha)\max_{s\in E} f(s)\right]. \tag{9.9}$$

10 EUA and CD-EUA

In this Appendix, we show the main differences between the notion of CD-EUA and the Equilibrium under Ambiguity (EUA) introduced by Eichberger and Kelsey (2014).

10.1 EUA

Let us first recall the equilibrium concept as defined in Eichberger and Kelsey (2014).

Consider a strategic game $\Gamma = (I, (S_i, u_i)_{i\in I})$. Players form beliefs about the opponents' strategy choice represented by convex capacities (see Appendix B). For a convex capacity μ_i on S_{-i}, the intersection of the supports of all probability distributions in the core of μ_i, which is always non-empty, provides a natural definition of its support.[51]

Definition 10.1. The support of a *convex* capacity μ_i on S_{-i}, denoted by $\text{supp}^C(\mu_i)$, is given by

$$\text{supp}^C(\mu_i) := \bigcap_{p\in \text{core}(\mu_i)} \text{supp}(p). \tag{10.1}$$

A JP-capacity $\nu_i^{JP}(\alpha_i, \mu_i)$ is a weighted average of μ_i and its dual, $\overline{\mu}_i$, that is,

$$\nu_i^{JP}(\alpha_i, \mu_i) := \alpha_i\mu_i + (1-\alpha_i)\bar{\mu}_i. \tag{10.2}$$

Noting that a convex capacity and its dual are simply two ways of representing the same information, the support of a JP-capacity is defined in terms of its convex part, μ_i.

Definition 10.2. The support of a JP-capacity $\nu_i^{JP}(\alpha_i, \mu_i)$ on S_{-i}, denoted by $\text{supp}^C(\nu_i^{JP})$, is given by

$$\text{supp}^C(\nu_i^{JP}) := \text{supp}^C(\mu_i)$$

The Choquet expected utility of a strategy $s_i \in S_i$ with respect to a JP-capacity $\nu_i^{JP}(\alpha_i, \mu_i)$ on S_{-i} based on beliefs given by the convex capacity μ_i is

$$V_i^{JP}(s_i \mid \alpha_i, \mu_i) = \alpha_i \min_{p\in\text{core}(\mu_i)} \int u_i(s_i, s_{-i})\, dp(s_{-i}) + (1-\alpha_i) \max_{p\in\text{core}(\mu_i)} \int u_i(s_i, s_{-i})\, dp(s_{-i}).$$

The notion of EUA requires equilibrium beliefs and behavior to be consistent. That is, players believe that their opponents play only strategies which are best replies. The

[51] Ryan (2002) studies different support notions for sets of probability distributions. For convex capacities, Eichberger and Kelsey (2014) show that most support notions in the literature coincide.

best-reply correspondence of player i with respect to a JP-capacity $\nu_i^{JP}(\alpha_i, \mu_i)$ is given by

$$R_i\big(\nu_i^{JP}(\alpha_i, \mu_i)\big) := \arg\max_{s_i \in S_i} V_i^{JP}(s_i \mid \alpha_i, \mu_i). \tag{10.3}$$

An EUA is a list of convex capacities (or JP-capacities) that satisfy the following consistency requirement: Each strategy combination in the support of each JP-capacity should consist of best replies of the opponent players with respect to their JP-capacities.

Definition 10.3. An n-tuple of convex capacities $\big(\hat{\mu}_1, \ldots, \hat{\mu}_n\big)$ constitutes an *Equilibrium in Beliefs Under Ambiguity* if for all players $i \in I$,

$$\varnothing \neq \operatorname{supp}^C(\hat{\mu}_i) \subseteq \underset{j \neq i}{\times} R_j\left(\nu_i^{JP}(\alpha_i, \hat{\mu}_i)\right). \tag{10.4}$$

Notice that the support of a convex capacity μ_i may be an empty set. Therefore, non-emptiness of the supports of the equilibrium JP-capacities must be additionally required.

10.2 CD-EUA

In this paper, we model players' beliefs over the opponents' strategy choice by belief functions which are a special case of convex capacities (see Appendix B). Moreover, we consider belief functions which admit a particular structure.

Our goal is to allow beliefs to be influenced by context information available to the players when playing a game $\Gamma = (I, (S_i, u_i)_{i \in I})$. Context information is considered to be exogenous reflecting (possibly partial and incomplete) knowledge about the likely strategy choices of players. Context information is modeled by a mass distribution β_i on Σ_{-i}. We combine this context information with the beliefs of player i about her opponents strategic behavior based on the knowledge of their payoffs. These beliefs are endogenous and are represented by a mass distribution σ_i on Σ_{-i} which is a probability distribution on $S(\Sigma_{-i})$, the set of single strategy combinations. For given β_i, σ_i and a degree confidence $\delta_i \in [0, 1]$, we define the mass distribution $\gamma_i(\sigma_i, \beta_i, \delta_i)$ on Σ_{-i} as a weighted average of σ_i and β_i. That is,

$$\gamma_i(\sigma_i, \beta_i, \delta_i) := (1 - \delta_i)\sigma_i + \delta_i \beta_i, \tag{10.5}$$

which defines a context-dependent belief function $\phi_i^{\gamma_i(\sigma_i, \beta_i, \delta_i)}$ on Σ_{-i} by

$$\phi_i^{\gamma_i(\sigma_i, \beta_i, \delta_i)}(E) = \sum_{A \subseteq E} \gamma_i(A \mid \sigma_i, \beta_i, \delta_i). \tag{10.6}$$

Hence, we consider belief functions which can be expressed as a convex mixture of a probability distribution σ and a belief function defined by an exogenous context-related mass distribution β_i. Modeling beliefs is this way offers a tractable way to study strategic ambiguity due to context-information and thus it may facilitate economic applications.

The context-dependent belief function $\phi_i^{\gamma_i(\sigma_i,\beta_i,\delta_i)}$ is a convex capacity and hence, we can built a JP-capacity. For a JP-capacity based on $\phi_i^{\gamma_i(\sigma_i,\beta_i,\delta_i)}$ and a degree of ambiguity attitude $\alpha_i \in [0,1]$, the CEU payoff of a strategy $s_i \in S_i$ takes the simple form:

$$V_i(s_i \mid \alpha_i, \gamma_i(\sigma_i,\beta_i,\delta_i)) = (1-\delta_i) \sum_{s_{-i}\in S_{-i}} u_i(s_i,s_{-i})\sigma_i(\{s_{-i}\}) \tag{10.7}$$
$$+ \delta_i \sum_{E\subseteq S_{-i}} \beta_i(E) \left[\alpha_i \min_{s_{-i}\in E} u_i(s_i,s_{-i}) + (1-\alpha_i) \max_{s_{-i}\in E} u_i(s_i,s_{-i})\right].$$

Hence, the best-reply correspondence of player i depends on σ_i, β_i and δ_i:

$$R_i(\sigma_i,\beta_i,\delta_i) := \arg\max_{s_i\in S_i} V_i\big(s_i \mid \alpha_i, \gamma_i(\sigma_i,\beta_i,\delta_i)\big). \tag{10.8}$$

A CD-EUA is a list of context-dependent belief functions that satisfy a similar consistency requirement as EUA. Players believe that their opponents play only strategies that are optimal with respect to their beliefs. However, as a support of a context-dependent belief function $\phi_i^{\gamma_i(\sigma_i,\beta_i,\delta_i)}$, we consider the support of σ_i, a probability distribution on S_{-i}. This leads us to the definition of CD-EUA which we repeat below for the sake completeness.

Definition 10.4 (**Definition 3.2**). An n-tuple of context-dependent belief functions

$$\left(\phi_1^{\gamma_1(\widehat{\sigma}_1,\beta_1,\delta_1)}, \ldots, \phi_n^{\gamma_n(\widehat{\sigma}_n,\beta_n,\delta_n)}\right)$$

is a *Context-Dependent Equilibrium Under Ambiguity* if for all players $i \in I$,

$$\text{supp}^C(\widehat{\sigma}_i) \subseteq \times_{j\neq i} R_j(\widehat{\sigma}_j,\beta_j,\delta_j). \tag{10.9}$$

Notice that $\widehat{\sigma}_i$ is a product probability distribution on S_{-i} and thus the usual support notion guarantees its non-emptiness, i.e., $s_{-i} \in \text{supp}^C(\widehat{\sigma}_i)$ if and only if $\widehat{\sigma}_i(\{s_{-i}\}) > 0$. Another advantage of $\widehat{\sigma}_i$ is that it allows us to study how the standard notion of strategic independence in interplay with context information may affect equilibrium behavior.

10.3 EUA and CD-EU: A detailed comparison

In the last two sections, we pointed out that EAU and CD-EUA use a similar consistency condition between equilibrium beliefs and equilibrium behavior as in Nash equilibrium, Players expect their opponents to choose strategies that are optimal for their beliefs. Both solution concepts, however, implement this consistency requirement differently.

Besides different types of equilibrium beliefs, another crucial difference is which parts of these beliefs are used to determine which strategy combinations are "expected" (i.e., the supports of equilibrium beliefs) and whether the "expected" behavior is also optimal. In EUA, all strategy combinations in the support of the convex capacity must be best replies. In CD-EUA, all strategy combinations in the support of the probability distribution that defines the context-dependent belief function must be best replies of the opponents.

A more detailed comparison between CD-EUA and EUA is very subtly. In general, in a CD-EUA, a player may believe that the opponents play strategies suggested by the context information. Context-related strategies, however, do no need to be be optimal. EAU cannot model such situations. To scrutinize this conceptual difference, let us restrict the EUA notion to context-dependent belief functions while maintaining the consistency requirement of Definition C.3. That is, let us consider an n-tuple of context dependent belief functions $(\phi_1^{\widehat{\gamma}_1(\sigma_1,\beta_1,\delta_1)}, \ldots, \phi_n^{\widehat{\gamma}_n(\sigma_n,\beta_n,\delta_n)})$ such that for all players $i \in I$,

$$\varnothing \neq \operatorname{supp}^C \left(\phi_i^{\widehat{\gamma}_i(\sigma_i,\beta_i,\delta_i)}\right) \subseteq \underset{j\neq i}{\times} R_j \left(\nu_j^{JP}\big(\alpha_j, \phi_j^{\widehat{\gamma}_j(\sigma_j,\beta_j,\delta_j)}\big)\right). \tag{10.10}$$

where $\nu_j^{JP}\big(\alpha_j, \phi_j^{\widehat{\gamma}_j(\sigma_j,\beta_j,\delta_j)}\big)$ denotes the JP-capacity based on $\phi_j^{\widehat{\gamma}_j(\sigma_j,\beta_j,\delta_j)}$. Call this version of EUA a restricted EUA. Notice that the endogenous component of equilibrium beliefs in the restricted EUA is the mass distribution $\widehat{\gamma}_i$ in contrast to $\widehat{\sigma}_i$ in a CD-EUA.

The support of a belief function is directly related to its mass distribution. This helpful result has been proven by Dominiak and Eichberger (2016, Proposition 3.2).

Lemma 10.1. *Let $\phi_i^{\gamma_i}$ be a belief function with mass distribution γ_i, then*

$$s_{-i} \in \bigcap_{p\in \operatorname{core}(\phi_i^{\gamma_i})} \operatorname{supp}(p) \quad \textit{if and only if} \quad \gamma_i(\{s_{-i}\}) > 0. \tag{10.11}$$

Hence, in order to check whether list of belief functions $(\phi_1^{\widehat{\gamma}_1(\sigma_1,\beta_1,\delta_1)}, \ldots, \phi_n^{\widehat{\gamma}_n(\sigma_n,\beta_n,\delta_n)})$ constitutes a restricted EUA, one needs to verify for the corresponding mass distributions $\big(\widehat{\gamma}_1(\sigma_1,\beta_1,\delta_1), \ldots, \widehat{\gamma}_n(\sigma_n,\beta_n,\delta_n)\big)$ whether for each player $i \in I$, it is true that

$$\widehat{\gamma}_i(\{s_{-i}\}) > 0 \ \text{ implies } s_{-i} \in \underset{j\neq i}{\times} R_j \left(\nu_j^{JP}\big(\alpha_j, \phi_j^{\widehat{\gamma}_j(\sigma_j,\beta_j,\delta_j)}\big)\right). \tag{10.12}$$

However, in a CD-EUA $\big(\phi_1^{\gamma_1(\widehat{\sigma}_1,\beta_1,\delta_1)}, \ldots, \phi_n^{\gamma_n(\widehat{\sigma}_n,\beta_n,\delta_n)}\big)$, for each player $i \in I$,

$$\widehat{\sigma}_i(\{s_{-i}\}) > 0 \ \text{ implies } s_{-i} \in \underset{j\neq i}{\times} R_j(\widehat{\sigma}_j, \beta_j, \delta_j). \tag{10.13}$$

Since $\gamma_i(\sigma_i, \beta_i, \delta_i) = (1-\delta_i)\,\sigma_i + \delta_i\beta_i$, each strategy combination $s_{-i} \in S_{-i}$ such that $\widehat{\sigma}_i(\{s_{-i}\}) > 0$ implies $\gamma_i(\{s_{-i}\}|\widehat{\sigma}_i, \beta_i, \delta_i) > 0$. The reverse is, however, not true in general, leading us to the following difference between a CD-EUA and a restricted EUA:

$$\operatorname{supp}^C(\widehat{\sigma}_i) \subseteq \operatorname{supp}^C(\phi_i^{\widehat{\gamma}_i(\sigma_i,\beta_i,\delta_i)}). \tag{10.14}$$

The difference is the weaker optimality requirement. In CD-EUA, all strategy combinations s_{-i} such that $\widehat{\sigma}_i(\{s_{-i}\}) > 0$ must be optimal but not for all s_{-i} such that $\gamma_i(\{s_{-i}\}|\widehat{\sigma}_i, \beta_i, \delta_i) > 0$. Hence, a CD-EUA is not necessarily a restricted EUA. There may be strategy combinations s_{-i} for which $\widehat{\sigma}_i(\{s_{-i}\}) = 0$ holds but $\gamma_i(\{s_{-i}\}|\widehat{\sigma}_i, \beta_i, \delta_i) > 0$ since the context information gives them a strictly positive weight $\beta_i(\{s_{-i}\}) > 0$. However, such strategy combinations need not be optimal thus ruling out a restricted EUA.

It is easy to see, however, that the following sufficient conditions will guarantee that any strategy combination in a CD-EUA is also in a restricted EUA with respect the context-dependent beliefs.

Lemma 10.2. *Consider a CD-EUA* $\left(\phi_1^{\gamma_1(\widehat{\sigma}_1,\beta_1,\delta_1)},\ldots,\phi_n^{\gamma_n(\widehat{\sigma}_n,\beta_n,\delta_n)}\right)$.

If either

(i) $\operatorname{supp}(\beta_i)=\emptyset$, *or*

(ii) $\operatorname{supp}(\beta_i)\subseteq \underset{j\neq i}{\times} R_j(\widehat{\sigma}_j,\beta_j,\delta_j)$

then $\gamma_i(\{s_{-i}\}|\widehat{\sigma}_i,\beta_i,\delta_i)>0$ implies $s_{-i}\in \underset{j\neq i}{\times} R_j(\widehat{\sigma}_j,\beta_j,\delta_j)$.

If exogenous context information either does not put positive weights on singletons or if singletons on which the exogenous context information puts strictly positive weights are best replies then $\gamma_i(\{s_{-i}\}|\widehat{\sigma}_i,\beta_i,\delta_i)>0$ implies that s_{-i} consists of best replies.

In terms of applications, the above conditions rule out a direct conflict between context information and endogenously determined equilibrium beliefs. Such a conflict can arise if and only if context information suggests that the opponents choose strategies which cannot be best replies. In Example 3.2, suppose context information would suggest Player 1 to choose T and Player 2 to choose R, i.e., $\beta_1(\{R\})=1$ and $\beta_2(\{T\})=1$. Then, the second condition in Lemma 10.2 would rule out such inconsistent context information.