Online Appendix:

Testing Behavioral Hypotheses in Signaling Games

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E Existence of Separating HTE

In this Appendix, we provide sufficiency conditions for existence of a separating Rational HTE and a separating Behaviorally Consistent HTE for each finite (monotone) signaling games in $\mathcal{G}_M$. For the sake of completeness of this appendix, we recall Conditions (i)-(iv), which define $\mathcal{G}_M$.

We assume that Θ, $\mathcal{M}$ and $\mathcal{A}$ are finite (partially ordered) sets of real numbers:

$$\begin{aligned} \Theta &= \{\theta_1, \theta_2, \ldots, \theta_T\} \quad \text{where } \theta_t \in \mathbb{R} \text{ for } t = 1, \ldots, T; \\ \mathcal{M} &= \{m_1, m_2, \ldots, m_L\} \quad \text{where } m_l \in \mathbb{R} \text{ for } l = 1, \ldots, L; \\ \mathcal{A} &= \{a_1, a_2, \ldots, a_K\} \quad \text{where } a_k \in \mathbb{R} \text{ for } k = 1, \ldots, K. \end{aligned}$$

For the Sender, we assume that u_S satisfies Monotonicity and Single-Crossing Property.

(i) (Monotonicity) $u_S(\theta, m, a)$ is strictly decreasing in m and strictly increasing in a for any θ.

(ii) (Single-Crossing Property) For each $a \in \mathcal{A}$, all $\theta, \theta' \in \Theta$ and $m, m' \in \mathcal{M}$ such that $\theta' > \theta$ and $m' > m$, $u_S(\theta, m, a) \leq u_S(\theta, m', a')$ implies $u_S(\theta', m, a) < u_S(\theta', m', a')$.

For the Receiver, we assume that her best-reply correspondence is message-independent, single-valued, and increasing in θ. Moreover, the "highest" type θ_T has an incentive to signal m_L.

(iii) For each $m \in \mathcal{M}$ and $\mu := \mu(\cdot \mid m) \in \Delta(\Theta)$, $BR(\mu, m) = BR(\mu)$. Moreover, $BR(\mu(\theta) = 1)$ is increasing in θ, and $BR(\mu(\theta) = 1)$ is single-valued for each $\theta \in \Theta$.

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(iv) For m_1, m_L and θ_T, $u_S(\theta_T, m_L, BR(\mu(\theta_T) = 1)) \geq u_S(\theta_T, m_1, BR(\mu(\theta_1) = 1))$.

As mentioned before, Conditions (i)-(iv) resemble the properties of monotone signaling games in the continuous case (see Mailath, 1987; Cho and Sobel, 1990; Kreps and Sobel, 1994).

E.1 Rational HTE

Since the message space $\mathcal{M}$ is a finite set, we need to assume that $\mathcal{M}$ is sufficiently rich to guarantee existence of a separating Rational HTE for each signaling game in $\mathcal{G}_M$. More precisely, $\mathcal{M}$ is said to be *rich* if for each $\theta \in \{\theta_1, \theta_2, \ldots, \theta_T\}$, the following optimization problem has a solution: For type θ_1, the optimization problem

$$\arg\max_{m \in \mathcal{M}} u_S(\theta_1, m, BR(\mu(\theta_1) = 1)), \tag{1}$$

has a solution, denoted by m_1^* (which is identical to m_1). For type θ_2, the optimization problem

$$\arg\max_{m \in \mathcal{M}} u_S(\theta_2, m, BR(\mu(\theta_2) = 1)), \tag{2}$$

$$\text{s.t. } u_S(\theta_1, m_1^*, BR(\mu(\theta_1) = 1)) = u_S(\theta_1, m, BR(\mu(\theta_2) = 1)),$$

has a solution, denoted m_2^*. Note that θ_2 strictly prefers m_2^* over m_1^* by Single-Crossing Property. For each $\theta_t \in \{\theta_3, \ldots, \theta_T\}$, the optimization problem

$$\arg\max_{m \in \mathcal{M}} u_S(\theta_t, m, BR(\mu(\theta_t) = 1)), \tag{3}$$

$$\text{s.t. } u_S(\theta_{t-1}, m_{t-1}^*, BR(\mu(\theta_{t-1}) = 1)) = u_S(\theta_{t-1}, m, BR(\mu(\theta_t) = 1)),$$

has a solution, denote by m_t^*.[1] Again, θ_t strictly prefers m_t^* over m_{t-1}^* by Single-Crossing Property.

This richness condition allows us to construct for each message m° off the path, a rational hypothesis that is consistent with m°, demonstrating that a separating Rational HTE exists.

Proposition 5 *If $\mathcal{M}$ is rich, then there exists a separating Rational HTE for each game in $\mathcal{G}_M$.*

Proof. Consider a strategy profile (b_S^*, b_R^*) such that

$$b_S^*(m_1^*|\theta_1) = 1,\ b_S^*(m_2^*|\theta_2) = 1, \ldots,\ b_S^*(m_T^*|\theta_T) = 1, \tag{4}$$

where $m_1^* < m_2^* < \ldots < m_T^*$ and

$$b_R^*(BR(\mu^*)|m) = 1 \text{ for each } m \in \mathcal{M}, \tag{5}$$

[1] We write m_t^* to denote the message sent by type θ_t.

where

$$
\begin{aligned}
\mu^*(\theta_1|m) &= 1 \text{ for } m \in \{m_1, m_2, \ldots, m_2^*\} \setminus \{m_2^*\}, \\
\mu^*(\theta_2|m) &= 1 \text{ for } m \in \{m_2^*, \ldots, m_3^*\} \setminus \{m_3^*\}, \\
&\vdots \\
\mu^*(\theta_T|m) &= 1 \text{ for } m \in \{m_T^*, \ldots, m_L\}.
\end{aligned} \tag{6}
$$

By the richness condition, (b_S^*, b_R^*, μ^*) constitutes a separating PBE. Hence, we need to show that there exits a separating HTE supporting the PBE. To this end, we will construct rational hypotheses that justify the PBE beliefs, $\mu^* = (\mu^*(\cdot|m)_{m\in\mathcal{M}})$.

First, we construct a rational hypothesis π_0 that justifies the posteriors on the equilibrium path. The Receiver's belief $\bar{\beta}_R$ such that $\bar{\beta}_R = b_S^*$ and the prior p induce

$$\pi_0 = \bar{\beta}_R(m|\theta)p(\theta) \text{ for any } (m, \theta) \in \mathcal{M} \times \Theta. \tag{7}$$

By applying Bayes' rule, we thus obtain $\mu_\rho(\theta_t|m_t^*) = 1$ for each $t \in \{1, \ldots, T\}$.

Next, we construct a rational hypothesis π_{m° for each out-of-equilibrium message $m^\circ \in \mathcal{M}^\circ$. We divide $\mathcal{M}^\circ$ into two parts, and consider two steps: In Step 1, we consider out-of-equilibrium messages $m^\circ \in \mathcal{M}^\circ$ such that $m^\circ < m_T^*$. In Step 2, we consider $\hat{m}^\circ \in \mathcal{M}^\circ$ such that $\hat{m}^\circ > m_T^*$.

Step 1. For any two messages on the path, m_t^* and m_{t+1}^*, fix m° such that $m_t^* < m^\circ < m_{t+1}^*$. Recall that the PBE belief, given m°, is $\mu^*(\theta_t|m^\circ) = 1$.

Consider a rational strategy b'_R for the Receiver:

$$b'_R(\cdot|m) = b_R^*(\cdot|m) \text{ for } m \neq m^\circ \text{ and } b'_R(\cdot|m^\circ) \in \Delta(\mathcal{A}), \tag{8}$$

such that

$$u_S(\theta_t, m_t^*, BR(\mu(\theta_t) = 1)) = \sum_a u_S(\theta_t, m^\circ, a) b'_R(a|m^\circ). \tag{9}$$

Since m_t^* and m_{t+1}^* solve the optimization problem (3) and by construction of $b'_R(\cdot|m^\circ)$, we have

$$\sum_a u_S(\theta_t, m^\circ, a) b'_R(a|m^\circ) = u_S(\theta_t, m_{t+1}^*, BR(\mu(\theta_{t+1}) = 1)), \tag{10}$$

Moreover, Single-Crossing Property implies that

$$\sum_a u_S(\theta, m^\circ, a) b'_R(a, m^\circ) < u_S(\theta, m_{t+1}^*, BR(\mu(\theta_{t+1}) = 1)) \text{ for any } \theta > \theta_t. \tag{11}$$

Similarly, since Equation (9) holds for $m_t^* < m^\circ$, Single-Crossing Property implies

$$u_S(\theta, m_t^*, BR(\mu(\theta_t) = 1)) > \sum_a u_S(\theta, m^\circ, a) b'_R(a, m^\circ) \text{ for any } \theta < \theta_t. \tag{12}$$

Thus, only θ_t would choose m° as a best response. Denote such a best-response strategy by b'_S.

The Receiver's belief $\bar{\beta}_R = b'_S$, together with p, induces the following rational hypothesis:

$$\pi_{m'}(m, \theta) = \bar{\beta}_R(m|\theta)p(\theta) \text{ for any } (m, \theta) \in \mathcal{M} \times \Theta. \tag{13}$$

Since π_{m° yields the posterior $\mu_\rho(\theta_t|m^\circ) = 1$ conditional on m°, we can justify the PBE posteriors for each $m^\circ < m_T^*$.

Step 2. Fix $\hat{m}^\circ$ such that $m_T^* < \hat{m}^\circ \leq m_L$. Recall that the PBE belief, given $\hat{m}^\circ$, is $\mu^*(\theta_T|\hat{m}^\circ) = 1$.

Consider a rational strategy b''_R for the Receiver:

$$b''_R(BR(\mu(\theta_T) = 1)|\hat{m}^\circ) = 1 \text{ and } b''_R(\cdot|m) \in \Delta(\mathcal{A}) \text{ for } m \neq \hat{m}^\circ,$$

such that

$$u_S(\theta_T, \hat{m}^\circ, BR(\mu(\theta_T) = 1)) = \sum_a u_S(\theta_T, m_1, a) b''_R(a|m_1). \tag{14}$$

Note that b''_R is well-defined by Condition (iv). Hence, type θ_T would choose either m_1 or m_T as a best response to b''_R. By Equation (14) and $m_1 < \hat{m}^\circ$, Single-Crossing Property implies that

$$u_S(\theta, \hat{m}^\circ, BR(\mu(\theta_T) = 1)) < \sum_a u_S(\theta, m_1, a) b''_R(a|m_1) \text{ for any } \theta < \theta_T. \tag{15}$$

Hence, only θ_T would choose $\hat{m}^\circ$ as a best response. Denote such a best-response strategy by b''_S.

The Receiver's belief $\bar{\beta}_R = b''_S$, together with p, induces the following rational hypothesis:

$$\pi_{\hat{m}^\circ}(m, \theta) = \bar{\beta}_R(m|\theta)p(\theta) \text{ for any } (m, \theta) \in \mathcal{M} \times \Theta. \tag{16}$$

Since $\pi_{\hat{m}^\circ}$ induces $\mu_\rho(\theta_T|\hat{m}^\circ) = 1$ conditional on $\hat{m}^\circ$, we can justify the PBE belief for each $\hat{m}^\circ > m_T^*$.

We can now choose a second-order prior ρ with $supp(\rho) = \{\pi_0, \pi_{m^\circ}\}_{m^\circ \in \mathcal{M}^\circ}$ such that

$$\{\pi_0\} := \arg\max_{\pi \in supp(\rho)} \rho(\pi) \text{ and } \{\pi_{m^\circ}\} := \arg\max_{\pi \in supp(\rho)} \rho_{m^\circ}(\pi) \text{ for each } m^\circ \in \mathcal{M}^\circ. \tag{17}$$

Hence, there exists a separating Rational HTE, $(b_S^*, b_R^*, \rho, \mu_\rho^*)$, supporting the PBE (b_S^*, b_R^*, μ^*). ■

E.2 Behaviorally Consistent HTE

For existence of a separating Behaviorally Consistent HTE, we need an additional assumption. Beside the richness condition, we need to assume that the equilibrium message m_T^* signaled by the "highest" type θ_T is the "highest" message in $\mathcal{M}$; i.e., $m_T^* = m_L$.[2]

Proposition 6 *If $\mathcal{M}$ is rich and $m_T^* = m_L$, then there exists a separating Behaviorally Consistent HTE for each game in $\mathcal{G}_M$.*

Proof. By Conditions (i) through (iv), as shown in the first part of the proof of Proposition 1, there exists a separating PBE, (b_S^*, b_R^*, μ^*) with $\mu^* = (\mu^*(\cdot|m)_{m\in\mathcal{M}})$.

By using the equilibrium strategy b_S^*, we construct a rational hypothesis π_0 that justifies the PBE beliefs on the path. That is, the Receiver's belief $\bar{\beta}_R$ such that $\bar{\beta}_R = b_S^*$ and p induce

$$\pi_0 = \bar{\beta}_R(m|\theta)p(\theta) \ \text{ for any } (m,\theta) \in \mathcal{M} \times \Theta. \tag{18}$$

By applying Bayes' rule, the initial hypothesis π_0 yields $\mu_\rho(\theta_t|m_t^*) = 1$ for each $t \in \{1, \ldots, T\}$.

Next, we construct a behaviorally consistent hypothesis π_{m° for each out-of-equilibrium message $m^\circ \in \mathcal{M}^\circ$. For any m_t^* and m_{t+1}^*, fix m° such that $m_t^* < m^\circ < m_{t+1}^*$. We use the Receiver's rational strategy b'_R constructed in the proof of Proposition 1 (see Equation (8) and (9)).

For any type $\theta \neq \theta_t$, the optimal of b_S^* is maintained. That is, each $\theta \neq \theta_t$ best responds to b'_R according to b_S^*. However, for type θ_t, both messages m_t^* and m° are best responses to b'_R, and so are the mixtures between m_t^* and m°. Hence, the Sender has many best-response strategies to b'_R. For each $\theta \in \Theta$ and $t \in \{1, \ldots, T\}$ and a parameter $\varepsilon \in [0, 1]$, consider the following strategy:

$$\begin{aligned} b'_S(\cdot|\theta) = b_S^*(\cdot|\theta) \quad &\text{if} \quad \theta \neq \theta_t, \\ b'_S(m_t^*|\theta) = (1-\varepsilon) \text{ and } b'_S(m^\circ|\theta) = \varepsilon \quad &\text{if} \quad \theta = \theta_t, \end{aligned}$$

For $\varepsilon^* \in (0, 1)$, the Receiver's belief $\bar{\beta}_R = b'_S(\varepsilon^*)$ and p induce the following rational hypothesis:

$$\pi_{m^\circ}(m,\theta) = \bar{\beta}_R(m|\theta)p(\theta) \ \text{ for any } (m,\theta) \in \mathcal{M} \times \Theta. \tag{19}$$

By applying Bayes' rule, π_{m° yields the PBE belief $\mu_\rho(\theta_t|m^\circ) = 1$ conditional on m°.

Note that π_{m° induces the PBE posteriors on the path, i.e.,

$$\mu_\rho(\theta_i|m_i) = 1 \ \text{ for each } \ m_i \in \{m_1^*, m_2^*, \ldots, m_T^*\},$$

[2]This condition is also necessary to build a behaviorally consistent hypothesis for m°. Suppose there is additional message m_{L+1}. To construct a behaviorally consistent hypothesis for m_{L+1}, there should be a rational hypothesis that induces $\mu(\theta_T|m_T^*) = 1$ and $\mu(\theta_T|m_{L+1}) = 1$. It requires that both m_T^* and m_{L+1} are best responses for type θ_T to a rational strategy for the Receiver. However, there is no such a rational strategy from the rational Receiver.

and the PBE posterior off the path, $\mu_\rho(\theta_t|m^\circ) = 1$ conditional on m°. Hence, π_{m° rationalizes the Receiver's equilibrium best response on the path. Thus, π_{m° is behaviorally consistent with π_0. Since m° is chosen arbitrarily, we can construct a behaviorally consistent hypothesis π_{m° for each $m^\circ \in \mathcal{M}^\circ$.

Finally, we can choose a second-order prior ρ with $supp(\rho) = \{\pi_0, \pi_{m^\circ}\}_{m^\circ \in \mathcal{M}^\circ}$ such that

$$\{\pi_0\} := \arg\max_{\pi \in supp(\rho)} \rho(\pi) \text{ and } \{\pi_{m^\circ}\} := \arg\max_{\pi \in supp(\rho)} \rho_{m^\circ}(\pi) \text{ for each } m^\circ \in \mathcal{M}^\circ. \tag{20}$$

Hence, there exists a separating Behaviorally Consistent HTE $(b_S^*, b_R^*, \rho, \mu_\rho^*)$. ■

References

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